Franz W. Peren

Math for Business and Economics

Compendium of Essential Formulas

Third Edition

Springer

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For my father, Paul.

Preface

Preface to the 3rd, revised and supplemented edition

The 3rd edition of this compendium of formulas for math for business and economics has been revised and supplemented partially. My always reliable research assistant Nawid Schahab has contributed to the current edition. He deserves my thanks. Should any mistakes remain, such errors shall be exclusively at the expense of the author. The author is thankful in advance to all users of this formulary for any constructive comments or suggestions.

Bonn, February 2024 Franz W. Peren

Preface to the 2nd, revised and supplemented edition

The 2nd edition of this compendium of formulas for math for business and economics has been revised and supplemented partially. My valuable research assistant Nawid Schahab has contributed to the current edition. He deserves my thanks. Should any mistakes remain, such errors shall be exclusively at the expense of the author. The author is thankful in advance to all users of this formulary for any constructive comments or suggestions.

Bonn, January 2023 Franz W. Peren

Preface to the 1st edition

The following book is based on the author's expertise in the field of business mathematics. After completing his studies in business administration and mathematics, he started his career working for a global bank and the German government. Later he became a professor of business administration, specialising in quantitative methods. He has been a professor at the Bonn-Rhein-Sieg University in Sankt Augustin, Germany since 1995, where he is mainly teaching business mathematics, business statistics, and operations research. He has also previously taught and conducted research at the University of Victoria in Victoria, BC, Canada and at Columbia University in New York City, New York, USA. To the author's best knowledge and beliefs, this formulary presents its mathematical contents in a practical manner, as they are needed for meaningful and relevant application in global business, as well as in universities and economic practice.

The author would like to thank his academic colleagues who have contributed to this work and to many other projects with creativity, knowledge and dedication for more than 25 years. In particular, he would like to thank Ms. Eva Siebertz and Mr. Nawid Schahab, who were instrumental in managing and creating this formulary. Special thanks are given to Ms. Camilla Demuth, Ms. Linh Hoang, and Ms. Michelle Jarsen.

Should any mistakes remain, such errors shall be exclusively at the expense of the author. The author is thankful in advance to all users of this formulary for any constructive comments or suggestions.

Bonn, March 2021 Franz W. Peren

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List of Abbreviations

ACT	Advance Corporation Tax
a.m.	ante meridiem
APR	Annual Percentage Rate
AU	area unit(s)
BCD	Binary Coded Decimal
BEP	break-even point
BGB	German Civil Code
bit	binary digit
calcul.	calculate, calculation
cf.	confer
CFPB	Consumer Financial Protection Bureau
cm	centimetre(s)
CM	Contribution Margin
const.	constant
c.p.	ceteris paribus
det	determinant
e	Euler’s number
EC	European Commission
ED	edition
e.g.	exempli gratia

etc.	et cetera
EU	European Union
gal.	gallon
gen.	general
h	hour(s)
ICMA	International Capital Market Association
i.e.	id est
incl.	includes, including
inf	infimum
int	integer function
IP	inflection point(s)
IRR	internal rate of return
ISDA	International Swaps and Derivatives Association
j	relative periodic interest rate
kbyte	kilobyte
kg	kilogram(s)
l	litre(s)
lb	pound(s)
lim	limit
ln	natural logarithm
log	logarithm
ltd.	limited

LU	length unit(s)
max	maximum, maximise
mbyte	megabyte
min	minimise, minimum
min(s)	minute(s)
mm	millimetre(s)
norm	normal
NPV	net present value
opt	optimise, optimisation
oz	ounce(s)
p.a.	per annum
p.m.	post meridiem
QU	quantity unit(s)
rad	radius
regen	regeneration
rep.	repetition
resp.	respectively
sgn	signum
SI	Système International d'Unités
S_S	solution set
S_U	universal set

sup	supremum
TU	time unit(s)
USP	unique selling proposition
U.S.	United States of America
w/	with
w/o	without
yd	yard(s)

Chapter 1

Mathematical Signs and Symbols

Remark: The signs and symbols are partly shown in applications. For definitions see dedicated passage.

1.1 Pragmatic Signs

$a \approx b$	a approximately similar to b
$a \ll b$	a small towards b, a can be neglected compared to b
$a \gg b$	a large towards b
$a \mathrel{\widehat{=}} b$	a equivalent to b, e.g. $1\,\text{cm} \mathrel{\widehat{=}} 10\,\text{mm}; 1\,\text{inch} \mathrel{\widehat{=}} 25.4\,\text{mm}$
$a \wedge b$	a and b
$a \vee b$	a or b
...	and so forth (until), omission

1.2 General Arithmetic Relations and Links

(a, b are figures, elements, objects)

$a = b$	a equals b, arithmetic fundamental term, identity
$a \neq b$	a unequal to b, no identity
$a := b$	a equals b by definition
$a < b$	a less than b, fundamental term, e.g. $-6 < -2$
$a > b$	a greater than b, e.g. $3 > -8$

F. W. Peren, *Math for Business and Economics*,
https://doi.org/10.1007/978-3-662-68860-1_1

$a \leq b$	a less than or (at most) equal to b, $a \leq 8$ is equivalent to $]-\infty, 8]$
$a \geq b$	a greater than or (at least) equal to b, is equivalent to $b \leq a$
$a + b$	a plus b, sum of a and b, arithmetic fundamental term
$a - b$	a minus b, difference between a and b, single-digit linking sign
$a \cdot b$	a times b, product of a and b, arithmetic fundamental term
$\frac{a}{b}$	a divided by b, quotient of a and b, e.g. $\frac{16}{4} = 16 \div 4 = 4$
$\sum_{i=1}^{n} a_i$	Sum over a_i of i equals 1 up to n, $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_n$
$\prod_{i=1}^{n} a_i$	Product over a_i of i equals 1 up to n, $\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot ... \cdot a_n$

1.3 Sets of Numbers

$\mathbb{N}$	set of natural numbers, $\mathbb{N} = \{0, 1, 2, ...\}$
$\mathbb{N}^*$	set of positive natural numbers, $\mathbb{N}^* = \mathbb{N} \backslash \{0\} = \{1, 2, 3, ...\}$
$\mathbb{Z}$	set of integers, $\mathbb{Z} = \{... -2, -1, 0, 1, 2, ...\}$
$\mathbb{Q}$	set of rational numbers, $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$
$\mathbb{Q}^*$	set of rational numbers which vary from zero, $\mathbb{Q}^* = \mathbb{Q} \backslash \{0\}$
$\mathbb{Q}^+$	set of positive rational numbers
$\mathbb{Q}_0^+$	set of positive rational numbers plus zero
$\mathbb{R}$	set of real numbers
$\mathbb{R}^*$	set of real numbers which vary from zero
$\mathbb{R}^+$	set of positive real numbers
$\mathbb{R}_0^+$	set of positive real numbers plus zero
$\mathbb{C}$	set of complex numbers
$]a, b[$	open interval from a to b $\{x \mid a < x < b\}$

$]a,\infty[$	open, unbounded interval starting at a, $\{x \mid a < x\}$
$[a,b]$	closed interval from a to b, $\{x \mid a \leq x \leq b\}$
$[a,\infty[$	closed, unbounded interval starting at $a, \{x \mid a \leq x\}$
$[a,b[$	left-closed, right-open interval from a to b, $\{x \mid a \leq x < b\}$

1.4 Special Numbers and Links

$(a,b \in \mathbb{R};\ n,m \in \mathbb{Z};\ s \in \mathbb{N})$

a^n	a to the power of n, n^{th} power of a for $n \geq 0$
$\sqrt{a} = a^{\frac{1}{2}} = b$	root (square root) of a, equivalent to $b^2 = a$ for $b \geq 0$, $a \geq 0$
$\sqrt[n]{a} = a^{\frac{1}{n}} = b$	n^{th} root of a, equivalent to $b^n = a$ for $b \geq 0$, $a \geq 0$
$n!$	n factorial, $n! = \prod_{i=1}^{n} a_i = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$
$\operatorname{sgn} a$	signum of a (algebraic sign), e.g. $\operatorname{sgn}(-3) = -1$
$\lvert a \rvert$	absolute value of a, e.g. $\lvert -8 \rvert = 8$
$a_{[i]}$	a in the i^{th} position; e.g. $5; 6; 7; a_{[2]} = 6$
∞	infinity, note: ∞ is not a number
π	$3.1415926...$
e	Euler's Number, $e = 2.718281$

1.5 Limit

$\lim\limits_{x \to 0} f(x) = a$	a is the limit of the function $f(x)$ for x towards 0, i.e. $x^{\,x \to 0}$ gradually approaches the value 0, the function's value $f(x)$ converges (limits) towards a

$\lim\limits_{x \to \infty} f(x) = b$	b is the limit of the function $f(x)$ for x towards ∞
$\lim\limits_{x \to 5} f(x) = c$	c is the limit of the function $f(x)$ for x towards 5

1.6 Exponential Functions, Logarithm

e^x	exponential function of x, e to the power of x
$\ln x$	natural logarithm of x to base e; $\log_e x = \ln x$
$\log_a x$	logarithm of x to base a; $\log_a x = y \Leftrightarrow a^y = x$ with x ; $a > 0$ and $a \neq 1$
$\log x$	common logarithm of x to base 10 $\log x = \lg x = \log_{10} x$
$\mathrm{lb}\, x$	binary (dyadic) logarithm of x to base 2 $\mathrm{lb}\, x = \log_2 x$

1.7 Trigonometric Functions, Hyperbolic Functions

$\sin\ x$	sine of x
$\cos\ x$	cosine of x
$\tan\ x$	tangent of x
$\cot\ x$	cotangent of x

$\sinh\ x$	hyperbolic sine of x
$\cosh\ x$	hyperbolic cosine of x
$\tanh\ x$	hyperbolic tangent of x
$\coth\ x$	hyperbolic cotangent of x

$\operatorname{arcsin} y$	arc sine of y
$\operatorname{arccos} y$	arc cosine of y
$\operatorname{arctan} y$	arc tangent of y
$\operatorname{arccot} y$	arc cotangent of y
$\operatorname{arsinh} y$	area hyperbolic sine of y
$\operatorname{arcosh} y$	area hyperbolic cosine of y
$\operatorname{artanh} y$	area hyperbolic tangent of y
$\operatorname{arcoth} y$	area hyperbolic cotangent of y

1.8 Vectors, Matrices

$a, b, x, y, ...$	signs for vectors, also $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{x}, \overrightarrow{y}$
$0, \overrightarrow{0}$	zero vector, identity element regarding addition of vectors
$\|a\| = a$	absolute value of a, $\|a\| = \sqrt{a \cdot a}$
$< (a, b)$	angle between a and b
$a \perp b$	a orthogonal to b
$a \times b$	a cross b
A, B, ...	signs for matrices

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} = (a_{ij})\ m,n\text{-matrix } A$$

element a_{ij} (i^{th} row, j^{th} column)

A'	transposed matrix for A $(A')' = A$

$E_{n\times n} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & & \dots & 1 \end{pmatrix}$	identity (unit) matrix; diagonal matrix, whose elements of the main diagonals are all 1 and whose remaining elements are all 0
$A_{n\times n}^{-1}$	inverse matrix for A, $A \cdot A^{-1} = E$
$r(A)$	rank of A, also $Rg(A)$

1.9 Sets

$\{a_1,...,a_n\}$	set with the elements a_1, ..., a_n
$a \in A$	a is element of A
$a \notin A$	a is not element of A e.g.: $3 \notin \{4,5,6\}$
$A \subset B$	A is proper subset of B, e.g. every element of A also belongs to B, but B contains at least one element that does not belong to A. For example: $A \subset B$ if $A = \{1;\ 2;\ 3;\ 4\}$ and $B = \{1;\ 2;\ 3;\ 4;\ 5;\ 6\}$
$A \subseteq B$	A ist subset of B, e.g. every element of A also belongs to B. This includes $A = B$. For example: $A \subseteq B$ if $A = \{1;\ 2;\ 3;\ 4\}$ and $B = \{1;\ 2;\ 3;\ 4\}$
$A \not\subset B$	A is not a proper subset of B, e.g. **not** every element from A also belongs to B and B contains at least one element that does not belong to A. For example: $A \not\subset B$ if $A = \{1;\ 2;\ 3;\ 7\}$ and $B = \{1;\ 2;\ 3;\ 4;\ 5;\ 6\}$
$A \cup B$	A union B, A **or** B, includes common elements
$A \cap B$	A intersection B, A **and** B, includes all occuring

	elements
$A\backslash B$	relative complement of A and B, A **not** B, e. g. $\{2,3,4\}\{2,4\} = \{3\}$
$\bar{B}$	complement of B, includes all elements, which are not included in B
$\emptyset = \{\}$	empty set, includes no elements

1.10 Relations

$(a,\, b)$	(ordered) pair of a and b, also $\langle a;\, b\rangle$
$A \times B$	A cross B, cartesian product of A and B, set of all (ordered) pairs from A and B

1.11 Functions

$f = f(x)$	f of x, f is a function dependent on x
$D_f;\ D(f)$	domain of f
$R_f;\ R(f)$	range of f, codomain of f
$f\colon A \to B$	f is a transformation of A into B

1.12 Order Structures

$\min X$	minimum of X, least element of X
$\max X$	maximum of X, greatest element of X
$\sup X$	supremum of X, least upper bound of X
$\inf X$	infimum of X, greatest lower bound of X

1.13 SI[1] Multiplying and Dividing Prefixes

d	deci	10^{-1}	da	deca	10^{1}
c	centi	10^{-2}	h	hecto	10^{2}
m	milli	10^{-3}	k	kilo	10^{3}
μ	micro	10^{-6}	M	mega	10^{6}
n	nano	10^{-9}	G	giga	10^{9}
p	pico	10^{-12}	T	tera	10^{12}
f	femto	10^{-15}	P	peta	10^{15}
a	atto	10^{-18}	E	exa	10^{18}
z	zepto	10^{-21}	Z	zetta	10^{21}
y	yocto	10^{-24}	Y	yotta	10^{24}

---

[1] SI is the abbreviation for an international unit system of physical quantities. The SI (Système International d'Unités) defines seven coherent basic units, which can be represented as products of powers. The SI units thus make it possible to define a certain dimension simply by adding one of the prefixes mentioned above, without adding an additional numerical factor. The SI units include the metre (m), the kilogram (kg), the second (s), the ampere (A), the kelvin (K), the mole (mol), and the candela (cd). Derived units, such as the newton (N), can also be formed using the algebraic relations.

1.14 Greek Alphabet

Name	Lower Case Letter	Upper Case Letter
alpha	α	A
beta	β	B
gamma	γ	$\varGamma$
delta	δ	$\varDelta$
epsilon	ε	ε
zeta	ζ	Z
eta	η	H
theta	θ	$\varTheta$
iota	ι	I
kappa	κ	K
lambda	λ	$\varLambda$
mu	μ	M
nu	ν	N
xi	ξ	$\varXi$
omicron	o	O
pi	π	$\varPi$
rho	ρ	P
sigma	σ	$\varSigma$
tau	τ	T
upsilon	υ	$\varUpsilon$
phi	ϕ	$\varPhi$
chi	χ	X
psi	ψ	$\varPsi$
omega	ω	$\varOmega$

Chapter 2

Logic

2.1 Mathematical Logic

$\neg\varphi \;,\; \bar{\varphi}$	negation	$\neg\varphi =$ not φ
$\varphi \wedge \psi$	conjunction	φ and ψ
$\varphi \vee \psi$	disjunction	φ or ψ
$\varphi \dot{\vee} \psi$	alternative	either φ or ψ, exclusionary or
$\varphi \Rightarrow \psi$	implication	φ implies ψ, ψ follows after φ, also written as $\varphi \rightarrow \psi$
$\varphi \Leftrightarrow \psi$	equivalence	φ equivalent to ψ, φ is similar to ψ, also written as $\varphi \leftrightarrow \psi$
$\varphi \not\Leftrightarrow \psi$	anticoincidence	negated equivalence, exclusionary either-or
$\varphi \leftarrow \psi$	replication	if; if ψ applies then φ follows
$\forall\, x$	universal quantifier	for all x (applies)
$\exists\, x$	existential quantifier	there is (at least) one x for which it applies

2.2 Propositional Logic

2.2.1 Propositional Variable

$a, b, \ldots$ are letters or other symbols which can be used as placeholders for *statements* or *truths*.

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https://doi.org/10.1007/978-3-662-68860-1_2

2.2.2 Truth Tables

a	b	$\neg a$	$a \wedge b$	$a \vee b$
t	t	f	t	t
t	f	f	f	t
f	t	t	f	t
f	f	t	f	f

with:
t = true
f = false

Symbol	Meaning
A	A is a statement that can be true (t) or false (f). **truth values** t (true); f (false) Examples: The statement “7 is a prime number” is *true*, the statement “$8-3=4$” is *false*, “$7x+4=25$” is only valid when “$x=3$”. “3” is called *solution*.
$v(A)$	$v(A)$ is referred to as the truth value of the statement A. $v(A)=1$ means that A is *true* and $v(A)=0$ means that A is *false*.
$\neg A$	The *negation* $\neg A$ (or $\bar{A}$) of the statement A is *true* when A is false and *false* when A is true.
$A \wedge B$	The *conjunction* $A \wedge B$ is *true* when both statements are true and *false* when there is at least one false statement.
$A \vee B$	The *disjunction* $A \vee B$ is *true* when there is at least one true statement, and *false* when both statements are false.

$A \Rightarrow B$	The *implication* $A \Rightarrow B$ means: When A is *true*, B is also *true*. A is considered as condition (*premise*) and B as consequence (*conclusion*). $A \Rightarrow B$ is *only false* when a false conclusion is drawn from a true premise.
$A \Leftrightarrow B$	The *equivalence* $A \Leftrightarrow B$ means: When A is *true*, B is also *true* and vice versa. $A \Leftrightarrow B$ is *only false* when one of the statements is true and the other one is false.
$\exists$	“There is” (e.g.: $\exists x \in \boldsymbol{\Theta} : x^2 = 4$ means: there is a rational number x with $x^2 = 4$).
$\forall$	“For all” (e.g.: $\forall x \in \boldsymbol{\Theta} : x^2 \geq 0$ means: for all rational numbers x with $x^2 \geq 0$).

Chapter 3

Arithmetic

3.1 Sets

3.1.1 General

Notation

$\{a_1, \ldots, a_n\}$	set with the elements $a_1, \ldots, a_n$
$\{x \mid A(x)\}$	quantity of all x, to which $A(x)$ applies
$\emptyset$, also $\{\}$	empty set, includes no elements (no elements included)
$a \in A$	a is element of A, $a, b \in A \Leftrightarrow a \in A \wedge b \in A$
$a \notin A$	a is not element of A, e.g. $3 \notin \{4, 5, 6\}$
$A = B$	A equals B (set with identical elements, i.e. set equality)
$A \subseteq B$	A is improper subset of B, also $A \subset B$
$A \subsetneq B$	A is proper subset of B, when: $A \subseteq B \wedge A \neq B$, proper inclusion relation "included and unequal"
$A \supseteq B$	A is superset of B
$A \cap B$	intersection of A and B, $A \cap B = \{x \mid x \in A \wedge x \in B\}$
$A \cup B$	set union of A and B, $A \cup B = \{x \mid x \in A \vee x \in B\}$
$A \setminus B$	relative complement of A and B, $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$ (read: A without B)
$\bar{A}$	complement set of A, $\bar{A} = G \setminus A$ (G is the universal set)

F. W. Peren, *Math for Business and Economics*,
https://doi.org/10.1007/978-3-662-68860-1_3

$A \times B$	product set of A and B, $A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$
$P(A)$	power set of A; $P(A) = \{T \mid T \subseteq A\}$ $P(A)$ is the set of all subsets T of A

Bounds, Limits of a Set

A universal set, $\mathbb{S}_{\mathrm{U}}$, is bounded upwards (or downwards) if it has at least one upper (or lower) bound B. If both conditions apply, $\mathbb{S}_{\mathrm{U}}$ is bounded:

$B \geq x \quad (B \leq x)$ with $x \in \mathbb{S}_{\mathrm{U}}$

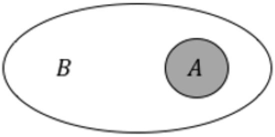
infimum:	$\inf x$	greatest lower bound, upper bound
supremum:	$\sup x$	least upper bound, lower bound

3.1.2 Set Relations

Inclusion

If A is a subset of B (superset), then each $a_i \in A$ is also $a_i \in B$

$A \subset B \Leftrightarrow B \supset A$ with $x \in A \Rightarrow x \in B$



$A \subset B$

Equality

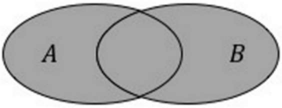
(Equivalence: “A equals B”)

$A = B$ with $x \; (x \in A \Leftrightarrow x \in B)$

3.1.3 Set Operations

Union of two sets $A \cup B$;
disjunction: “A or B”

$A \cup B = \{x \mid x \in A \vee x \in B\}$

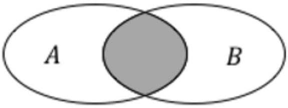


$A \cup B$

Intersection of two sets $A \cap B$;
conjunction: “A and B”

A and B are **conjunct** for: $A \cap B = \{x \mid x \in A \wedge x \in B\}$

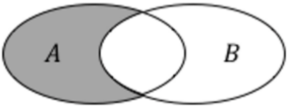
A and B are **disjoint** for: $A \cap B = \emptyset$



$A \cap B$

Relative complement of two sets $A \setminus B$, “A without B”

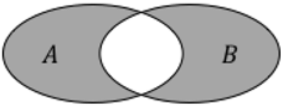
$A \setminus B = \{x \mid x \in A \wedge x \notin B\}$



$A \setminus B$

Symmetric difference of A and B

$A \triangle B = (A \cup B) \setminus (A \cap B)$

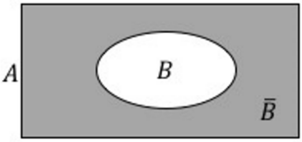


$A \Delta B$

Complement of the set B

Set of all elements,
which are not included in B

$\bar{B} = \{x \mid x \in A \wedge x \notin B\}$



$\bar{B} = A \backslash B$

Power set of B

Set of all subsets of a set B

$P(B) = \{x \mid x \subseteq B\}$ always valid: $\emptyset \in P(B)$ and $B \in P(B)$

Product (cartesian) of two sets $A \times B$, "A cross B"

$A \times B$ (product of two sets) is the set of **all** ordered pairs of elements (a, b) with $a \in A$ and $b \in B$

$A \times B = \{(a, b) \mid a \in A;\ b \in B\}$ $\quad A \times B \neq B \times A$

The product set $A_1 \times A_2 \times ... \times A_n$, $n \geq 1$, is the set of all ordered k-tuples ($x_1, ..., x_n$) of the elements x_1 of A_1, x_2 of A_2, ... , x_n of A_n.

3.1.4 Relations, Laws, Rules of Calculation for Sets

$S =$ universal set

Idempotent law

$$A \cup A = A$$

$$A \cap A = A$$

Commutative law

$$A \cap B = B \cap A$$

$$A \cup B = B \cup A$$

Associative law

$$(A \cap B) \cap C = A \cap (B \cap C)$$

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Absorption law

$$A \cap (A \cup B) = A$$

$$A \cup (A \cap B) = A$$

Distributive law

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup \emptyset = A \qquad A \cup S = S$$

$$A \cap \emptyset = \emptyset \qquad A \cap A = A \qquad A \cap S = A$$

$$A \setminus A = \emptyset \qquad A \setminus \emptyset = A$$

Product relations

$$(A \cup B) \times C = (A \times C) \cup (B \times C)$$
$$(A \cap B) \times C = (A \times C) \cap (B \times C)$$

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$
$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$(A \setminus B) \times C = (A \times C) \setminus (B \times C)$$
$$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$$

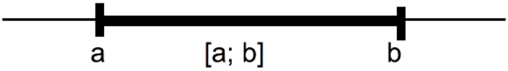
$$(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$$
$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

$$A \times B = \emptyset \Leftrightarrow A = \emptyset \vee B = \emptyset$$

$$A \subseteq C \wedge B \subseteq D \Rightarrow A \times B \subseteq C \times D$$

3.1.5 Intervals

An interval is a contiguous subset of real numbers, which is limited by two bounds ($=$ boundary points of the number line) a and b, $a < b$ for all $a, b \in \mathbb{R}$



- closed interval $[a, b] = \{x \mid a \leq x \leq b\}$
- open interval $]a, b[= \{x \mid a < x < b\}$
- half-open intervals $[a, b[= \{x \mid a \leq x < b\}$

  $]a, b] = \{x \mid a < x \leq b\}$
- infinite (half-open) intervals $\infty; -\infty$ are "improper numbers" in $\mathbb{R}$ with $-\infty < a$; $a < \infty$ for all $a \in \mathbb{R}$

$$[a;\infty[= \{x \mid a \leq x\}$$

$$]a;\infty[= \{x \mid a < x\}$$

$$]-\infty;a] = \{x \mid x \leq a\}$$

$$]-\infty;a[= \{x \mid x < a\}$$

3.1.6 Numeral Systems

decimal	dual/binary	BCD[1]	octal	hexadecimal
0	0000	0000 0000	0	0
1	0001	0000 0001	1	1
2	0010	0000 0010	2	2
3	0011	0000 0011	3	3
4	0100	0000 0100	4	4
5	0101	0000 0101	5	5
6	0110	0000 0110	6	6
7	0111	0000 0111	7	7
8	1000	0000 1000	10	8
9	1001	0000 1001	11	9
10	1010	0001 0000	12	A
11	1011	0001 0001	13	B
12	1100	0001 0010	14	C
13	1101	0001 0011	15	D
14	1110	0001 0100	16	E
15	1111	0001 0101	17	F
16	10000	0001 0110	20	10
17	10001	0001 0111	21	11
18	10010	0001 1000	22	12
19	10011	0001 1001	23	13
20	10100	0010 0000	24	14
etc.	etc.	etc.	etc.	etc.

---

[1] BCD ("binary-coded decimal") reads the pseudo-decimal numbers. The octal system uses the base 8, the hexadecimal system the base 16.

3.1.6.1 Decimal System (Decadic System)

Decimal powers: 10^k, $k \in \mathbb{Z}$

$10^0 = 1$

$10^1 = 10$ $10^{-1} = 0.1$

$10^2 = 100$ $10^{-2} = 0.01$

etc.

Decimal notation of an integer b $(k, n \in \mathbb{N})$

$$b = \pm \sum_{k=0}^{n} b_k \cdot 10^k = \pm \left(b_0 10^0 + b_1 10^1 + b_2 10^2 + \dots + b_{n-1} 10^{n-1} + b_n 10^n \right)$$

Base figures $b_k \in \{0, 1, 2, \dots, 9\}$

3.1.6.2 Dual System (Binary System)

1 bit (“binary digit”) symbolizes a “yes - no” decision.

1 byte $=$ 8 bit

1 kbyte $=$ 2^{10} byte $=$ 1.024 byte

1 mbyte $=$ 2^{10} kbyte $=$ 1.024 kbyte

etc.

Base symbols: 0.1

Place value: powers of 2

$$\sum_{k=-\infty}^{n} a_k \cdot 2^k \; a_k = 0.1$$

3.1.6.3 Roman Numeral System

Base symbols: $I = 1;\ V = 5;\ X = 10;\ L = 50;\ C = 100;$
$D = 500;\ M = 1000$

Notation: It starts on the left with the symbol of the largest number; the symbols I, X, C are written no more than three times; if the symbol of a smaller number precedes that of a larger number (e.g. $IV = 4$), its value is subtracted from the larger one, however this is only valid for CM, XC, IX, IV.

Example: 1998 is equivalent to $MCMXCVIII$ ($MIIM$ is not valid).

3.2 Elementary Calculus

3.2.1 Elementary Foundations

Fundamental arithmetic operations for $a, b, c \in \mathbb{R}$

		a	b	c
addition	$a+b=c$	summand	summand	sum
subtraction	$a-b=c$	minuend	subtrahend	difference
multiplication	$a \cdot b = c$	factor	factor	product
division	$\frac{a}{b} = c$	dividend, numerator	divisor, denominator	quotient, fraction

3.2.1.1 Axioms

Commutative law $a+b=b+a$ $a \cdot b = b \cdot a$

Associative law $(a+b)+c = a+(b+c)$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Distributive law $a \cdot (b+c) = a \cdot b + a \cdot c$

Sign conventions $(+a) \cdot (+b) = (-a) \cdot (-b)$ $a, b > 0$

$(+a) \cdot (-b) = (-a) \cdot (+b)$

$$\frac{(+a)}{(+b)} = \frac{(-a)}{(-b)} = +\frac{a}{b} \qquad \frac{(+a)}{(-b)} = \frac{(-a)}{(+b)} = -\frac{a}{b}$$

3.2.1.2 Factorisation

Remark: multiplication and division before addition and subtraction

$$a+(b+c-d) = a+b+c-d \qquad a-(b+c-d) = a-b-c+d$$

$$ac+bc = c \cdot (a+b) \qquad ac-bc = c \cdot (a-b) \qquad -ac-bc = -c \cdot (a+b)$$

$$a \cdot (b-c) = ab-ac \qquad a \cdot (b+c) = ab+ac$$

$$(a+b) \cdot (c+d) = ac+ad+bc+bd \qquad (a-b) \cdot (c-d) = ac-ad-bc+bd$$

$$(a+b) \cdot (c-d) = ac-ad+bc-bd \qquad (a-b) \cdot (c+d) = ac+ad-bc-bd$$

3.2.1.3 Relations

$$a < b \quad \Leftrightarrow \quad b > a \quad \Leftrightarrow \quad (b-a) > 0$$

$a < b$ and $c > 0$

$$\Rightarrow a + c < b + c$$

$$\Rightarrow a \cdot c < b \cdot c$$

$a < b$ and $a > 0$

$$\Rightarrow -a > -b$$

$$\Rightarrow \frac{1}{a} > \frac{1}{b}$$

3.2.1.4 Absolute Value, Signum

Definitions:

	absolute value of $a\,(\lvert a \rvert)$	signum of $a\,(\operatorname{sgn} a)$
$a > 0$	$\lvert a \rvert = +a$	$\operatorname{sgn} a = 1$
$a = 0$	$\lvert a \rvert = 0$	$\operatorname{sgn} a = 0$
$a < 0$	$\lvert a \rvert = -a$	$\operatorname{sgn} a = -1$

Laws:

$$|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$$

$$|a+b| \leq |a| + |b|$$
$$|a+b| \geq |a| - |b|$$

$$\operatorname{sgn}(a \cdot b) = \operatorname{sgn} a \cdot \operatorname{sgn} b; \qquad \operatorname{sgn}\left(\frac{a}{b}\right) = \frac{\operatorname{sgn} a}{\operatorname{sgn} b} \quad \text{with } b \neq 0$$

3.2.1.5 **Fractions** (for $a, b, c, d \in \mathbb{Z}$, denominator is always unequal to zero)

Reciprocal: $\frac{a}{b}$ is the reciprocal of $\frac{b}{a}$

a is the reciprocal of b, if $b = \frac{1}{a}$

Zero does not have a reciprocal; $\frac{1}{0}$ is not defined

Equality: $\frac{a}{b} = \frac{c}{d} \Leftrightarrow a \div b = c \div d \Leftrightarrow a \cdot d = b \cdot c$

Expand: $\frac{a}{b} = \frac{a \cdot z}{b \cdot z}$

Addition: $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

Multiplication: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

Simplification: $\frac{a \cdot z}{b \cdot z} = \frac{a}{b}$

Subtraction: $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$

3.2.1.6 Polynomial Division

1. The goal is to determine the zeros of a 3^{rd} degree polynomial.

2. The function is ordered such that the powers of the independent variables decrease from left to right.

3. The first zero is obtained by testing (trial and error) an integer divisor of the absolute term of the function.

4. 1st term dividend divided by 1st term divisor
   ⇒ 1. term quotient.

5. Reverse multiplication with the divisor.

6. Subtraction: 1st term dividend minus 1st term quotient; 2nd term dividend minus 2nd term quotient, if necessary, etc. The result of each subtraction is supplemented by the following term of the dividend. This process is iterated until an (additive) remainder is left that ideally results in zero (see example 2).

<u>Example 1:</u>

$$\begin{array}{l}
\quad (8x^2y - 6xy \;\; + 3x) \;\; \div (2xy+y) = 4x-5+\dfrac{3x+5y}{2xy+y} \\
-\,\underline{(8x^2y + 4xy)\quad} \\
\qquad\qquad (-10xy + 3x) \\
\qquad\quad -\,\underline{(-10xy - 5y)\quad} \\
\qquad\qquad\qquad\quad 3x+5y
\end{array}$$

<u>Example 2:</u>

$$\begin{array}{l}
\quad (t^3 \;-\; 8t^2 \;+\; 19t \;-\; 12) \div (2t-2) = \dfrac{1}{2}t^2 - \dfrac{7}{2}t + 6 \\
-\,\underline{\;(t^3 \;-\; t^2)\;} \qquad\quad \downarrow \qquad\;\; \downarrow \\
\qquad\quad -7t^2 + 19t \\
\qquad -\,\underline{(-7t^2 + 7t)} \qquad\quad \downarrow \\
\qquad\qquad\qquad (12t \;-\; 12) \\
\qquad\qquad\;\; -\,\underline{\;(12t \;-\; 12)\;} \\
\qquad\qquad\qquad\qquad\quad 0
\end{array}$$

3.2.1.7 Horner's Scheme (Horner's Method)[2]

1. The goal is to determine the zeros of a 3rd degree polynomial.

2. The function is ordered such that the powers of the independent variables decrease from left to right.

3. The first zero is obtained by testing (trial and error) an integer divisor of the absolute term of the function.

4. The coefficients are entered in the header of a table.

5. The first zero identified (by trial and error) is inserted in the header column.

6. In the bottom row, the coefficient before the independent variable is carried forward with the highest power.

7. This coefficient is multiplied by the value of the first zero and noted under the next coefficient with the second highest power.

8. The top row is added to the second row.

9. The value resulting from this addition is again multiplied by the zero first identified (by trial and error) and written under the coefficient with the next lower power.

10. The next steps are analogous, i.e. iterative.

11. As a result, the coefficients of the 2nd degree polynomial can now be read off directly. This 2nd degree polynomial can now be calculated, for example, by using the p/q formula.

---

[2] William George Horner (1786 - 1837) was an English mathematician.

Example:

$f(x) = 5x^3 - 8x^2 - 27x + 18$

	5	-8	-27	18	coefficients of the 3rd degree polynomial
$x = -2$		-10	36	-18	$-10 = -2 \cdot 5;\ 36 = -2 \cdot -18;$ $-18 = -2 \cdot 9$
	5	-18	9	0	coefficients of the 2nd degree polynomial

$\Rightarrow f(x) = 5x^2 - 18x + 9 = 0$

$5x^2 - 18x + 9 = 0$ divide by 5 to insert into p/q formula.

$x^2 - 3.6x + 1.8 = 0$ insert into p/q formula.

$p/q-$Formel

$x_1 = 3 \qquad x_2 = 0.6$ zeros are (-2|0), (0,6|0) and (3|0).

Remark:

If the highest exponent of the function to be solved is a four, the Horner's scheme must be applied successively twice.

3.2.2 Conversions of Terms

3.2.2.1 Binomial Formulas $(a, b \in \mathbb{R})$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)(a-b) = a^2 - ab + ab - b^2 = a^2 - b^2$$

3.2.2.2 Binomial Theorem $(a+b)^n$ with $a, b \in \mathbb{R}, n \in \mathbb{N}$.

for several values of n:

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a-b)^3 &= a^3-3a^2b+3ab^2-b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
(a-b)^4 &= a^4-4a^3b+6a^2b^2-4ab^3+b^4 \\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \\
(a-b)^5 &= a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5
\end{aligned}$$

3.2.2.3 General Binomial Theorem for Natural Exponents $(n \in \mathbb{N})$

$$\begin{aligned}
(a+b)^n &= \binom{n}{0}a^n+\binom{n}{1}a^{n-1}b+\ldots+\binom{n}{n-1}ab^{n-1}+\binom{n}{n}b^n \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \qquad \text{with} \quad a, b \in \mathbb{R}
\end{aligned}$$

3.2.2.4 General Binomial Theorem for Real Exponents $(\alpha \in \mathbb{R})$

$$(a+b)^\alpha = \binom{\alpha}{0}a^\alpha+\binom{\alpha}{1}a^{\alpha-1}b+\binom{\alpha}{2}a^{\alpha-2}b^2+\ldots+\binom{\alpha}{\alpha}b^\alpha$$

with $a, b, \alpha \in \mathbb{R}$
condition of convergence $|b| < |a|$

3.2.2.5 Polynomial Terms with $a,b,c,d \in \mathbb{R}$

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$$

$$(a+b+c)^3 = a^3+b^3+c^3+3\left(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b\right)+6abc$$

$$(a+b)\cdot(c+d-e) = ac+ad-ae+bc+bd-be$$

3.2.3 Summation and Product Notation

Index of summation or multiplication; $x \in \mathbb{Z}$

3.2.3.1 Summation Notation

$$\sum_{i=1}^{n} x_i = x_1+x_2+\ldots+x_n$$ read: “sum over x_i of $i = 1$ to n”

$$\sum_{i=1}^{n+1} x_i = \left(\sum_{i=1}^{n}\right)+x_{n+1}$$

$$\sum_{i=m}^{n} = x_m+x_{m+1}+\ldots+x_{n-1}+x_n$$ with $m < n$

$$\sum_{i=1}^{n} x_i = x_1$$

$$\sum_{i=1}^{0} x_i = 0$$

Rules:

$$\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1} nb_i \qquad n > 1$$

$$\sum_{i=1}^{n} a_i \cdot b_i \neq \sum_{i=1}^{n} a_i \cdot \sum b_i \qquad n > 1$$

$$\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i \qquad c = \text{constant}$$

$$\sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i = \sum_{i=1}^{n} a_i, \qquad m < n$$

$$\sum_{i=1}^{n} c = n \cdot c$$

$$\sum_{i=1}^{n} \sum_{j=i}^{m} a_{ij} = \begin{array}{l} a_{11} + a_{12} + \ldots + a_{1m} \\ + a_{21} + a_{22} + \ldots + a_{2m} \\ + \ldots \\ + a_{n1} + a_{n2} + \ldots + a_{nm} \end{array}$$

In general: $\sum_{i=m}^{n} a_i \cdot b_i \neq \sum_{i=m}^{n} a_i \cdot \sum_{i=m}^{n} b_i$

3.2.3.2 Product Notation

$$\prod_{i=1}^{n} x_i = x_1 \cdot x_2 \cdot \ldots \cdot x_{n-1} \cdot x_n$$

read: “product of all x_i for $i = 1$ to n”

$$\prod_{i=1}^{n+1} x_i = \left(\prod_{i=1}^{n} x_i \right) \cdot x_{n+1}$$

$$\prod_{i=m}^{n} x_i = x_m \cdot x_{m+1} \cdot \ldots \cdot x_{n-1} \cdot x_n$$

$$\prod_{i=1}^{1} x_i = x_1$$

$$\prod_{i=1}^{0} x_i = 1$$

Rules:

$\prod_{i=1}^{n}(a_i \cdot b_i) = \prod_{i=1}^{n} a_i \cdot \prod_{i=1}^{n} b_i$ also applies for division.

$\prod_{i=1}^{n}(c \cdot a_i) = c^n \cdot \prod_{i=1}^{n} a_i$ $c =$ constant

$\prod_{i=1}^{m} a_i \cdot \prod_{i=m+1}^{n} a_i = \prod_{i=1}^{n} a_i$ $m < n$

$\prod_{i=1}^{n} c = c^n$

3.2.4 Powers, Roots

Definitions $(a \in \mathbb{R}; n \in \mathbb{N})$

$a^n = a \cdot a \cdot a \cdot \ldots \cdot a$ $(n$ factors$)$

$a^1 = a$

$a^0 = 1$ $a \neq 0$

0^0 is not defined

reciprocal: $a^{-1} = \frac{1}{a}$ with $a \neq 0$

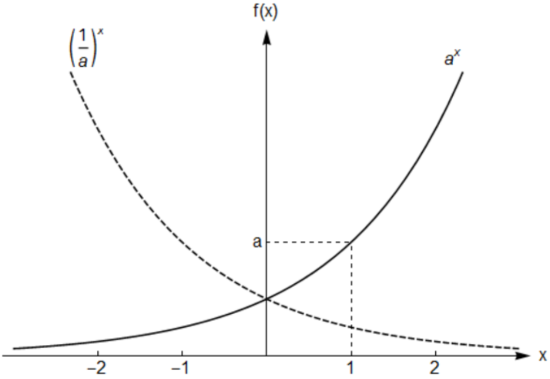
Sign Conventions $n \in \mathbb{Z}$

$$a > 0 \quad \Rightarrow \quad a^n > 0$$

$$a < 0 \quad \Rightarrow \quad \begin{cases} a^{2n} > 0 \\ a^{2n+1} < 0 \end{cases}$$

special: $(-1)^{2n} = 1$

$(-1)^{2n+1} = -1$



$a =$ constant; $a \in \mathbb{R}_0^+$

$n; m \in \mathbb{Z}^*$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{n}{m}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

Digression:

$\sqrt[n]{a} = x \quad \Leftrightarrow \quad x^n = a \qquad a =$ radicand $\quad n =$ order of the root

$$\sqrt[2]{a} = \sqrt{a}$$

$$\sqrt{a^2} = |a|$$

$$\sqrt[n]{0} = 0$$

$$\sqrt[n]{1} = 1$$

Theorems: with $(a,b,p,q \in \mathbb{R},\ m,n \in \mathbb{Z})$

$$a^m \cdot a^n = a^{m+n} \qquad a^n \cdot b^n = (a\cdot b)^n \qquad \frac{a^m}{a^n} = a^{m-n}$$

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n ; b \neq 0 \qquad (a^m)^n = a^{m\cdot n} \qquad pa^n \pm qa^n = (p \pm q)\cdot a^n$$

$$\sqrt[n]{\sqrt[m]{a}} = \sqrt[m]{\sqrt[n]{a}} = \sqrt[m\cdot n]{a} \qquad \sqrt[n]{a}\cdot\sqrt[n]{b} = \sqrt[n]{a\cdot b} \qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

$$\sqrt[n\cdot k]{a^{m\cdot k}} = \sqrt[n]{a^m} \qquad \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m = a^{\frac{m}{n}}$$

Rationalisation of the Denominator

If there is an algebraic function ($=$ root with argument) in the denominator of a fraction, it may under certain circumstances make sense to extend the fraction in such a way that the denominator becomes rational.

Examples:

$$\frac{4\cdot x}{\sqrt[3]{x^2}} = \frac{4\cdot x}{\sqrt[3]{x^2}}\cdot\frac{\sqrt[3]{x}}{\sqrt[3]{x}} = \frac{4\cdot x\sqrt[3]{x}}{\sqrt[3]{x^3}} = \frac{4\cdot x\cdot\sqrt[3]{x}}{x} = 4\cdot\sqrt[3]{x}$$

$$\frac{2}{3\cdot(a+\sqrt{b})} = \frac{2}{3\cdot(a+\sqrt{b})}\cdot\frac{(a-\sqrt{b})}{(a-\sqrt{b})} = \frac{2\cdot(a-\sqrt{b})}{3\cdot(a+\sqrt{b})\cdot(a-\sqrt{b})} =$$

$$= \frac{2\cdot(a-\sqrt{b})}{3\cdot(a^2-b)}$$

3.2.5 Logarithms

Definition:

The logarithm of b (numerus) to the base a is the real number c (exponent).

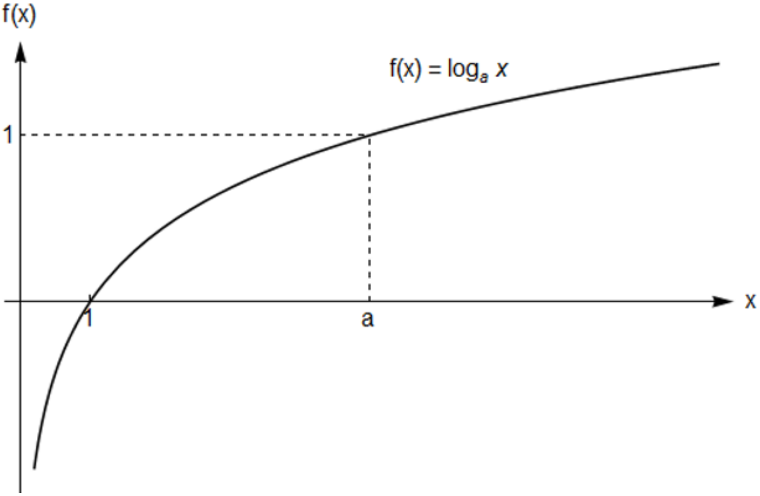
$$\log_a b = c \Leftrightarrow b = a^c \qquad a, b \in \mathbb{R}^+, a \neq 1$$

Every equation $a^x = b$ has exactly one real solution.

Rules: $\log_a a = 1$ $\quad \log_a (a^b) = b$ $\quad b \in \mathbb{R}$

$\log_a 1 = 0$ $\quad \log_a x < 0$ for $x < 1$

$\log_a x > 0$ for $x > 1$



Examples:

1) $3^x = 81$ $\qquad$ $x = \log_3 81 = 4$

test: $3^4 = 81$

2) $\log_5 0.008 = -3$ $\qquad$ test: $5^{-3} = 0.008$

3) $\log_{253} 100 = 0.8323$ $\qquad$ test: $253^{0.8323} = 100$

Logarithmic Laws $(a,u,v \in \mathbb{R}^+, a \neq 1)$

1) $\log_a(u \cdot v) = \log_a u + \log_a v$

2) $\log_a \frac{1}{u} = \log_a u^{-1} = -\log_a u$

3) $\log_a u^r = r \cdot \log_a u \qquad r \in \mathbb{R}$

4) $\log_a \sqrt[r]{u} = \log_a \left(u^{\frac{1}{r}}\right) = \frac{1}{r}\log_a u$

Logarithmic Systems: Common Logarithm

Base $a = 10$

Notations: $\log_{10} b = \lg b$

$\lg b = c \quad \Leftrightarrow \; b = 10^c$

$\lg 10^k = k \qquad k \in \mathbb{R}$

Natural Logarithm

Base $a = e$

$e =$ Euler's number $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.718281828459$

Notations: $\log_e b = \ln b$ $\ln =$ "logarithmus naturalis"

$\ln b = c$ $\Leftrightarrow\ b = e^c, \quad b > 0$

$\ln e^c = c$ $c \in \mathbb{R}$

Logarithm to an Arbitrary Base

Notations: $\log_a k = \dfrac{\log_k}{\log_a} = \dfrac{\ln_a}{\ln_k}$

Example: $\log_{4711} 15 = \dfrac{\log_{15}}{\log_{4711}} = \dfrac{\ln_{15}}{\ln_{4711}} = 0.3202$

3.2.6 Factorial

$n! = 1\cdot 2\cdot 3\cdot \ldots \cdot n = \prod\limits_{i=1}^{n} i$ $n \in \mathbb{N}^*$ (read: n factorial)

Recursion formula: $(k+1)! \quad = k!\cdot(k+1)$ $k \in \mathbb{N}$

Definitions: $0! = 1$ $1! = 1$

3.2.7 Binomial Coefficient (read “n choose k”)

For $n,k \in \mathbb{N}$:

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{for } 0 \leq k \leq n \\ 0 & \text{for } 0 \leq k \leq n \end{cases}$$

For $n,k \in \mathbb{N}$:

$$\binom{0}{0} = \binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{n}{1} = \binom{n}{n-1} = n$$

Pascal’s Triangle for Determining the Binomial Coefficients

$$\begin{array}{lcccccccccccll}
n=0 & & & & & & 1 & & & & & & \text{row sum} & 2^0 \\
n=2 & & & & 1 & & 2 & & 1 & & & & & 2^2 \\
n=3 & & & 1 & & 3 & & 3 & & 1 & & & & 2^3 \\
 & & & \diagdown & + & \diagup\;\diagdown & + & \diagup\;\diagdown & + & \diagup & & & & \\
n=4 & & 1 & & 4 & & 6 & & 4 & & 1 & & & 2^4 \\
n=5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & 2^5 \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & &
\end{array}$$

The boundary values are always 1, the mean values are the sum of the values immediately above them (left and right).

Examples:

$$\binom{7}{5} = \frac{7!}{5! \cdot (7-5)!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5) \cdot (1 \cdot 2)} = 21$$

$$\binom{-\frac{1}{3}}{2} = \frac{\left(-\frac{1}{3}\right) \cdot \left(-\frac{1}{3} - 1\right)}{2!} = \frac{\left(-\frac{1}{3}\right) \cdot \left(-\frac{4}{3}\right)}{1 \cdot 2} = \frac{2}{9} = 0.\overline{2}$$

3.3 Sequences

3.3.1 Definition

A sequence a_k is a mapping of natural numbers, $k \in \mathbb{N}^*$ (possibly also $k \in \mathbb{N}$) to a universal set $\mathbb{S}_{\text{U}}$, $a_k \in \mathbb{R}$:

$$a_k = a_1, a_2, a_3, ..., a_k \qquad k \in \mathbb{N}^*; a_k \in \mathbb{R}$$

If S corresponds to a set of points, a so-called point sequence is created; if S corresponds to a set of numbers, a so-called numeric sequence is created.

A real numeric sequence is an ordered set of real numbers. It corresponds to a discrete function of the mapping:

$a_k = f(k)$ with $D_f = \mathbb{N}^*$ and $R_f = \mathbb{R}$

Sequences can be finite or infinite.

Finite sequences have a last term a_n:

$a_k = a_1, ..., a_3$ with $a_i = 0$ for all $i > n$

Infinite sequences have an unlimited number of terms:

$a_k = a_1,\ a_2,\ ...$

Examples:

(1) $a_k = k^3$

$\Rightarrow a_k = 1, 8, 27, 64, 125, ...$

$a_5 = 5^{th}$ term $= 125$

(2) $a_k = (-1)^k \cdot (a_k + 1)$

$\Rightarrow a_k = -2, 3, -4, 5, -6, ...$

$a_5 = -6$

(3) Sequences with alternating signs:

$(3a)\quad a_k = (-1)^{k+1} = +1, -1, +1, -1, ...$

$(3b)\quad a_k = (-1)^k = -1, +1, -1, +1, ...$

Fundamental Terms:

A numeric sequence a_k is called

negatively definite $a_k < 0$

monotonically increasing	$a_k \leq a_k + 1$
strictly monotonically increasing	$a_k < a_k + 1$
monotonically decreasing	$a_k \geq a_k + 1$
strictly monotonically decreasing	$a_k > a_k + 1$
bounded above ($B_u =$ upper bound)	$a_k \leq B_u; B_u \in \mathbb{R}$
bounded below ($B_l =$ lower bound)	$a_k \geq B_l; B_l \in \mathbb{R}$
bounded	$B_l \leq a_k \leq B_u$
constant	$a_k = a_{k+1}$

Supremum, Infimum, Limits

The supremum of an upwards bounded sequence a_k, $\sup a_k$, is the least upper bound ($=$ the upper limit) of a_k.

Example:

$a_k = -k^3$

$$\Rightarrow a_k = -1, -8, -27, -64, -125, ...$$

Possible upper bounds are e.g. 17 or 0 or -1.
However, the supremum ($=$ the upper limit) of a_k is definitely:
$\sup a_k = -1$

The infimum of a downwards bounded sequence a_k, $\inf a_k$, is the greatest lower bound ($=$ the lower limit) of a_k.

Example:

$a_k = k^3$

$$\Rightarrow a_k = 1, 8, 27, 64, 125, \ldots$$

Possible lower bounds are e.g. -100 or 0 or 1.
However, as infimum ($=$ the lower limit) only one definite value exists:
$\inf a_k = 1$

3.3.2 Limit of a Sequence

The sequence a_k is called convergent with the limit g, if for any real, positive number ε nearly all sequence terms a_k lie within the ε-range of g, $]g-\varepsilon;\ g+\varepsilon[$:

$|a_k - g| < \varepsilon$ for nearly all $k \in \mathbb{N}^*$; $\varepsilon \in \mathbb{R}^+$

$\lim\limits_{k\to\infty} a_k = g$ or $\underset{k\to\infty}{a_k} \to g$

Read: The limit of a_k for k towards infinity is equal to g.

If a numerical sequence a_k has the limit g, a_k is called convergent, a_k converges towards g. If no limit exists, a_k is divergent.

Theorems:

For $\lim\limits_{k\to\infty} a_k = g_1$ and $\lim\limits_{k\to\infty} b_k = g_2$ applies:

(1) $\lim\limits_{k\to\infty} (a_k \pm b_k) = g_1 \pm g_2$

(2) $\lim\limits_{k\to\infty} (a_k \cdot b_k) = g_1 \cdot g_2$

(3) If, apart from the starting terms, all $b_k \neq 0$ and $g^2 \neq 0$, the following applies: $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = \dfrac{g_1}{g_2}$

(4) $\lim\limits_{k\to\infty} (a_k^n) = g_1^n \quad n \in \mathbb{N}^*$

(5) Every convergent sequence is limited.
Remark: Not every limited sequence is convergent.
The limited sequence $-1, +1, -1, +1, ...$ for example is divergent.

(6) Every limited and monotone sequence is convergent.

(7) $a_k \leq b_k \Rightarrow g_1 \leq g_2$

Null Sequence

A sequence a_k is called null sequence, if its limit is zero:

$$\lim_{k\to\infty} a_k = 0$$

Example:

$a_k = \dfrac{1}{k}$ is a null sequence for $k \in \mathbb{N}^*$, in that case: $\lim\limits_{k\to\infty} a_k = 0$

Improper Limit

a_k diverges towards ∞ or $-\infty$:

$$\lim_{k\to\infty} a_k = \infty \qquad \lim_{k\to-\infty} a_k = -\infty$$

Examples of Limits of Selected Numerical Sequences ($k \in \mathbb{N}^*$)

$$\lim_{k\to\infty} a_k = 0 \quad \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e = 2.718281828459...$$ (number)

$$\lim_{k\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k}-\ln k\right) = C = 0.57721$$ (Euler’s constant)

$$\lim_{k\to\infty}\frac{k!}{k^k\cdot e^{-k}\cdot\sqrt{k}} = \sqrt{2\pi}$$ (Stirling’s formula)

$$\lim_{k\to\infty}\sqrt[k]{a} = 1 \qquad a > 0 \qquad \lim_{k\to\infty}\sqrt[k]{k} = 1$$

3.3.3 Arithmetic and Geometric Sequences

Arithmetic Sequences

In an arithmetic sequence, the difference d between each two consecutive terms of a sequence a_k is constant:

$a_{k+1} - a_k = d$ with $d =$ constant

for all $k \in c,$

the numerical sequence is arithmetical.

Geometric Sequence

In a geometric sequence, the quotient q between each two consecutive terms of a sequence a_k is constant:

$$\frac{a_{k+1}}{a_k} = q \qquad \text{with} \qquad q = \text{constant}$$

$$\text{for all} \quad k \in \mathbb{N}^*,$$

the numerical sequence is geometrical.

3.4 Series

3.4.1 Definition

A series s_n (to a sequence a_k) is equivalent to the n^{th} partial sum of the first n terms (summands) of the sequence a_k:

$$s_n = a_1 + a_2 + ... + a_n = \sum_{k=1}^{n} a_k$$

3.4.2 Arithmetic and Geometric Series

Arithmetic Series

$$s_n = a_1 + a_2 + a_3 + ... + a_n = \sum_{k=1}^{n} a_k \qquad \text{with} \qquad \begin{aligned} d &= a_n - a_{n-1} = ... \\ &= a_3 - a_2 \\ &= a_2 - a_1 = \text{constant} \end{aligned}$$

k^{th} term: $a_k = a_1 + (k-1) \cdot d$

last term: $a_n = a_1 + (n-1) \cdot d$

sum: $s_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}[2a_1 + (n-1) \cdot d]$

Arithmetic Series of Higher Order

An arithmetic series of the i^{th} order is present if only the i^{th} difference sequence has constant terms:

$$a_k = b_i(k-1)^i + b_{i-1}(k-1)^{i-1} + ... + b_0 \qquad \text{with: } k = 1, ...$$

<u>Example:</u>

$a_k \;=\; 1 \quad 5 \quad 10 \quad 18 \quad 31 \quad 51 \quad ...$ basic sequence

$\triangle_1 a_k \;=\; 4 \quad 5 \quad 8 \quad 13 \quad 20 \quad ...$ 1st difference sequence

$\triangle_2 a_k \;=\; 1 \quad 3 \quad 5 \quad 7 \quad ...$ 2nd difference sequence

$\triangle_3 a_k \;=\; 2 \quad 2 \quad 2 \quad ...$ 3rd difference sequence

The primary sequence a_k is equivalent to an arithmetic series of the 3rd order.

Geometric Series

A geometric series of the i^{th} order is present if the quotient q of two consecutive terms is constant.

For $|q| < 1$: $$\sum_{n=0}^{n} q_0\, q^k = \qquad k = 0 \qquad = \frac{q_0}{1-q}$$

$$\text{with} \quad q = \frac{a_n}{a_{n-1}} = \frac{a_{n-1}}{a_{n-2}} = ... = \frac{a_3}{a_2} = \frac{a_2}{a_1} = \text{constant}$$

Example:

$a_o = 5;\ a_1 = 15,\ a_2 = 45,\ a_3 = 135,\ \ldots$

$$\Rightarrow \qquad \sum_{k=0}^{n} 5 \cdot 3^k = 5 \cdot 3^0 + 5 \cdot 3^1 + 5 \cdot 3^2 + 5 \cdot 3^3 + \ldots$$

$$= 5 + 15 + 45 + 135 + \ldots = \infty$$

with $q = 3$

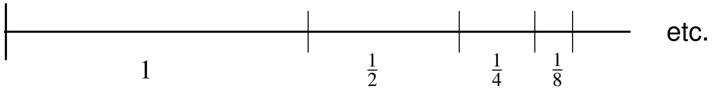
because $\frac{15}{5} = 3;\ \frac{45}{15} = 3$ etc.

Infinite Geometric Series

$s = 1 + x + x^2 + x^3 + \ldots + x^n$

$s = \frac{1 - x^{n+1}}{1 - x}$ with $x \neq 1$

Example:



$$s = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = \frac{1 - \frac{1}{2^{n+1}}}{0.5}$$

if $n \to \infty$, the following is valid:

$$\lim_{n\to\infty} s = \frac{1}{0.5} = 2,$$

i.e. in this example, the number 2 will never be reached.

Chapter 4

Algebra

4.1 Fundamental Terms

Variables are placeholders (e.g. $a, b, x, y,$ ...) that can be replaced by numbers from a given universal set $\mathbb{S}_\mathsf{U}$.

A **term** within a universal set $\mathbb{S}_\mathsf{U}$ is an expression composed of variables, numbers and/or arithmetic symbols. Division by zero is not possible.

Examples:

(1) 6

(2) $8-2$

(3) $x+2$

(4) $3b+5$

(5) x^2-4x+6 with $a,\ b,\ x,\ y, \in \mathbb{R}$

Equations and Inequations

An equation (inequation) is created when terms are connected by the equal sign "$=$" (the not-equal signs "$<, \leq, >, \geq$" or "$\neq$").

The **solution set** $\mathbb{S}_\mathsf{S}$ of an equation/inequation is the set of elements that makes the initial form a true statement when used in place of the variables. The equation/inequation has no solution if the solution set is equal to the empty set.

F. W. Peren, *Math for Business and Economics*,
https://doi.org/10.1007/978-3-662-68860-1_4

Examples:

(1) $(x-5)(x-3) = 0 \qquad x \in \mathbb{R} \qquad \mathbb{S}_{\mathrm{S}} = \{5;3\}$

(2) $x+1 \leq x \qquad x \in \mathbb{R} \qquad \mathbb{S}_{\mathrm{S}} = \emptyset$

Universal Equations

If a solution set $\mathbb{S}_{\mathrm{S}}$ is identical to the universal set $\mathbb{S}_{\mathrm{U}}$, the equation is generally valid with respect to the universal set $\mathbb{S}_{\mathrm{U}}$.

Examples:

(1) $2(x+1) = 2x+2 \qquad x \in \mathbb{R}$

(2) $(a+b)^2 = a^2+2ab+b^2 \qquad a,b \in \mathbb{R}$

Equivalent Transformations of Equations

Equations are equivalent if their solution sets are identical. With non-equivalent transformations (square/multiply/divide by terms containing the variable(s)) other solution sets can arise. Sample offered!

Examples: $x \in \mathbb{R}$

(1)

$$\begin{aligned} & 5 \cdot (x-1) && = 30 \\ \Leftrightarrow\quad & x-1 && = 6 && \text{(equivalent transformation)} \\ \Leftrightarrow\quad & x && = 7 && \mathbb{S}_{\mathrm{S}} = 7 \end{aligned}$$

(2)

$$\begin{aligned} & x-2 && = \sqrt{x} \\ \Rightarrow\quad & x^2-4x+4 && = x && \text{(no equivalent transformation)} \\ \Leftrightarrow\quad & x^2-5x+4 && = 0 \end{aligned}$$

$$\Leftrightarrow \quad x_{1,2} = \left(\frac{5}{2}\right) \pm \sqrt{\left(\frac{5}{2}\right)^2 - 4} \quad (p/q \text{ formula})$$

$$\Leftrightarrow \quad x_1 = 2.5 + \sqrt{(2.5)^2 - 4} = 4$$

$$\Leftrightarrow \quad x_2 = 2.5 - \sqrt{(2.5)^2 - 4} = 1$$

Sample offered!

$\Leftrightarrow$ 4 is the only solution of equation (2).

$\Leftrightarrow$ $\mathbb{S}_\text{S} = \{4\}$

4.2 Linear Equations

4.2.1 Linear Equations with One Variable

Normal form: $ax + b = 0$; $x \in \mathbb{R}$ $a \neq 0$

Equivalent transformations are used to separate the correct variable.

Example:

$$\begin{aligned} 50x + 40 &= -10x \\ \Leftrightarrow \quad 60x &= -40 \\ \Leftrightarrow \quad x &= -\frac{2}{3} \qquad \mathbb{S}_\text{S} = \left\{-\frac{2}{3}\right\} \end{aligned}$$

Fractional Equations

The domain of definition corresponds to the base set excluding the values where the denominator becomes zero.

Example:

$$\begin{aligned} \frac{5}{x-3} &= \frac{2}{x-1} && D=\mathbb{R}\backslash\{1;3\} \\ \Leftrightarrow\ 5\cdot(x-1) &= 2\cdot(x-3) \\ \Leftrightarrow\ x &= -\frac{1}{3} && \mathbb{S}_{\mathrm{S}}=\left\{-\frac{1}{3}\right\} \end{aligned}$$

Fractional Inequations with One Variable

For the transformation of fractional inequations, a case distinction must take place which is provided by the domain of definition. Definition gaps can accordingly be divided into cases that are to be investigated separately.

Example: $\frac{x-2}{x+3}<2$

Step 1: Defining the domain of definition

$D_f=\mathbb{R}\setminus\{-3\}$

Step 2: Case distinction, determined by the domain:

Case 1 positive denominator $x>-3$

Case 2 negative denominator $x<-3$

Step 3: Solving inequations separately

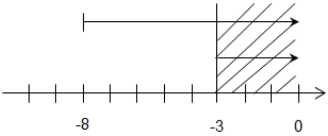
Case 1: $x > -3$

$$\begin{aligned} \frac{x-2}{x+3} &< 2 && \mid \cdot(x+3) \\ x-2 &< 2\cdot(x+3) && \\ x-2 &< 2x+6 && \mid -x;\ -6 \\ -8 &< x && \end{aligned}$$

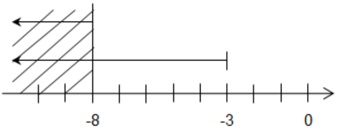
Case 2: $x < -3$

$$\begin{aligned} \frac{x-2}{x+3} &< 2 && \mid \cdot(x+3) \\ x-2 &> 2\cdot(x+3) && \mid \text{inversion of the inequality sign as multiplied with a negative number} \\ x-2 &> 2x+6 && \mid -x;\ -6 \\ -8 &> x && \end{aligned}$$

Step 4: Define intersections



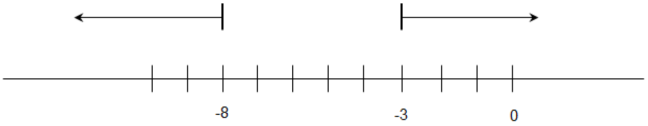
Intersection: $x > -3$



Intersection: $x < -8$

Step 5: Define union of sets/solution set

$x > -3 \quad \cup \quad x < -8$



$$\mathbb{S}_{\mathrm{S}} = \{x \,|\, x < -8 \cup x > -3\}$$

4.2.2 Linear Inequations with One Variable

When multiplying with or dividing by a negative number, the relation sign is reversed.

Example:

$$\begin{aligned} & -3x - 10 < 2 \cdot (x + 20) & \\ \Leftrightarrow \; & -3x - 10 < 2x + 40 & \quad | -2x; +10 \\ \Leftrightarrow \; & -5x < 50 & \quad | \div(-5) \\ \Leftrightarrow \; & x > -10 & \quad | \text{ inversion of the relation sign} \\ \Leftrightarrow \; & \mathbb{S}_{\mathrm{S}} = \{x \mid x > -10\} =]-10; \infty[& \end{aligned}$$

4.2.3 Linear Equations with Multiple Variables

A definite determination of n variables is only possible if n independent equations exist (unambiguously determinable equation system). If there are only r independent equations with n variables ($r < n$), then $(n - r)$ variables exist as free parameters and thus an infinite number of number tuples as solutions.

Examples:

(1) $3x + 8y = 100 \qquad x, y \in \mathbb{R}$

$$\Leftrightarrow \quad x = \frac{100 - 8y}{3} \text{ resp. } y = \frac{100 - 3x}{8}$$

A definite solution is not possible.

(2)

$$\begin{array}{rrcll}
(a) & 3x + 8y & = & 100 & \\
(b) & x + 2y & = & 50 & \mid -2y \\
\Leftrightarrow & x & = & 50 - 2y & \mid (b) \text{ resolved to } x \\
(b) \text{ in } (a) & 3 \cdot \underbrace{(50 - 2y)}_{x} + 8y & = & 100 & \\
\Leftrightarrow & y & = & -25 & \\
y \text{ in } (b) & x & = & 50 - 2 \cdot \underbrace{(-25)}_{y} & \\
\Leftrightarrow & x & = & 100 &
\end{array}$$

There is a definite solution, namely:

$$\mathbb{S}_S = \{(x, y) \mid x = 100; y = -25\}$$

4.2.4 Systems of Linear Equations

A **system of linear equations** consists of several linear equations. Its solution set is the set of all (ordered) tuples of values for which all equations become true statements.

Equivalent Transformations of Systems of Linear Equations

(1) Multiplication of an equation by a real number,

(2) Addition of the multiple of one equation to another equation (linear combination),

(3) Swapping equations.

Solving Systems of Linear Equations

(a) Substitution Method

An equation is solved for a variable. This term is inserted into another equation in the place of the corresponding variable.

Example:

$$\begin{array}{rlll}
(a) & x+2y & = 15 & \\
(b) & 2x-2y & = 24 & \mid +2y;\ \div 2 \\
\Leftrightarrow & x & = 12+y & \mid\ (b) \text{ resolved to } x
\end{array}$$

$$\begin{array}{rlll}
(b) \text{ in } (a) & \underbrace{12+y}_{x}+2y & = 15 & \mid -12 \\
\Leftrightarrow & 3y & = 3 & \mid \div 3 \\
\Leftrightarrow & y & = 1 &
\end{array}$$

replacing the y value in (b), results in the x value:

$$\begin{array}{rll}
y \text{ in } (b) & x & = 12+\underbrace{1}_{y}
\end{array}$$

$$\Leftrightarrow \quad x = 13$$

$$\mathbb{S}_{\text{S}} = \{(x,y) \mid x = 13; y = 1\}$$

(b) Equalisation Method

Two equations are solved for the same variable and the terms on the right sides are set equal.

Example:

$$\begin{aligned}
(a) \quad y &= -3x + 900 \\
(b) \quad y &= x + 200
\end{aligned}$$

equalling (a) to (b):

$$\begin{aligned}
(a) = (b) \quad x + 200 &= -3x + 900 && \mid +3x; -200 \\
\Leftrightarrow \quad 4x &= 700 && \mid \div 4 \\
\Leftrightarrow \quad x &= 175
\end{aligned}$$

replacing the x value in (a) or (b) results in the y value:

$$\begin{aligned}
x \text{ in } (b) \quad y &= \underbrace{175}_{x} + 200 && \Leftrightarrow y = 375 \\
\text{or} \quad x \text{ in } (a) \quad y &= -3 \cdot \underbrace{175}_{x} + 900 && \Leftrightarrow y = 375
\end{aligned}$$

$$\mathbb{S}_{\text{S}} = \{(x,y) \mid x = 175; y = 375\}$$

(c) Addition Method

Two equations are multiplied with or divided by suitable real numbers in such a way that one variable is eliminated by adding or subtracting the two equations.

Example:

$$\begin{array}{llcllrl}
(a) & 3x & + & 2y & = & 15 & \\
(b) & x & - & y & = & 12 & |\cdot(-3) \\
\hline
(a) & 3x & + & 2y & = & 15 & \\
(b) & -3x & + & 3y & = & -\ 36 & \\
\hline
(a)+(b) & 0x & + & 5y & = & -\ 21 & |\div 5 \\
 & y & & & = & 4.2 &
\end{array}$$

Replacing the y value in (a) or (b) results in the x value:

$$\begin{array}{lll}
y \text{ in } (a) & x = \dfrac{15-2\cdot(-4.2)}{3} & = 7.8 \\
y \text{ in } (b) & x = 12+(-4.2) & = 7.8
\end{array}$$

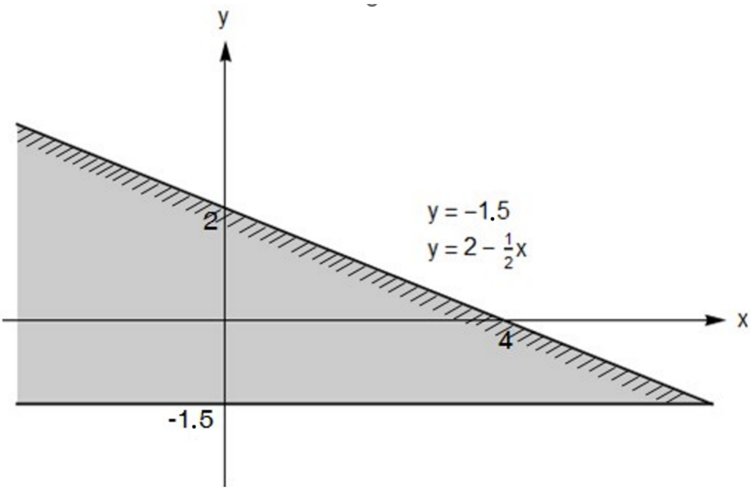
$$\mathbb{S}_{\mathsf{S}} = \{(x,y) \mid x = 7.8; y = -4.2\}$$

4.2.5 Linear Inequations with Multiple Variables

The solution set is the average (of the intersection) of the solution sets of the individual inequations.

<u>Example:</u>

$$\begin{array}{lllllll} & x+2y-4 & < 0 & \wedge & y \geq -1.5 & & x,y \in \mathbb{R} \\ \Leftrightarrow & y & < 2-\dfrac{1}{2}x & \wedge & y \geq -1.5 & & \end{array}$$



The solution set includes all points of the Cartesian coordinate system bounded by the straight lines $y = 2 - 0.5x \wedge y = -1.5$.

If the (boundary) line itself is excluded (relation signs $<$, $>$, $\neq$), it has to be drawn dashed (as in the example above).

4.3 Non-linear Equations

4.3.1 Quadratic Equations with One Variable

$a-b-c$ Formula

General form: $a^2+bx+c=0 \qquad a,\, b,\, c \in \mathbb{R} \qquad a,\, b \neq 0$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \quad = \quad x_{1,2} = \frac{-\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - 2ac}}{a}$$

p/q Formula

Normal form: $x^2+px+q=0$ (right side is equal to 0

absolute term of x^2 is equal to 1)

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} \qquad (p/q \text{ formula})$$

$\mathbb{S}_{\mathsf{S}} = \{x_1;\, x_2\}$

If the radicand is negative, the following applies: $\mathbb{S}_{\mathsf{S}} = \{\,\}$

Example:

$$2x^2-8x=-6 \qquad \text{(initial equation)}$$

$$\Rightarrow x^2 \underbrace{-4}_{p}\, x \underbrace{+3}_{q} = 0 \qquad \text{(normal form)}$$

$$\Rightarrow p=-4; \quad q=3$$

$$x_{1,2} = -\left(\frac{-4}{2}\right) \pm \sqrt{\left(\frac{-4}{2}\right)^2 - 3}$$

$$x_{1,2} = 2 \pm \sqrt{4-3}$$

$$\Rightarrow x_1 = 2+1 = 3$$

$$x_2 = 2-1 = 1 \qquad\qquad \mathbb{S}_{\mathrm{S}} = \{3;-1\}$$

Completing the Square

Normal form: $x^2+px+q=0$ (right side is equal to 0
absolute term of x^2 is equal to 1)

$$x^2+px = -q$$

Both sides are completed "quadratically", i.e. with a summand which arises from the first binomial form:

$$(a+b)^2 = a^2+2ab+b^2$$

b^2 is added, whereby b is obtained from the second summand:

$$px \mathrel{\hat{=}} 2ab$$

$$\Rightarrow \quad x \mathrel{\hat{=}} a \wedge p \mathrel{\hat{=}} 2b$$

$$\Leftrightarrow \quad b = \frac{p}{2}$$

$$\underbrace{x^2}_{a^2} + \underbrace{px}_{2ab} + \underbrace{\left(\frac{p}{2}\right)^2}_{b^2} = -q + \left(\frac{p}{2}\right)^2$$

Complementary term

$$\Leftrightarrow \quad \left(x+\frac{p}{2}\right)^2 = -q+\left(\frac{p}{2}\right)^2 \qquad \text{(first binomial form)}$$

$$\Rightarrow \quad x_{1,2}+\frac{p}{2} = \pm\sqrt{\left(\frac{p}{2}\right)^2-q}$$

$$\Leftrightarrow \quad x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

(corresponds to p/q formula)

Example:

$$\Leftrightarrow \quad 5x^2 - \frac{15}{2}x = 10$$

(initial equation)

$$\Leftrightarrow \quad x^2 - \frac{3}{2}x - 2 = 0$$

(normal form)

$$\Leftrightarrow \quad x^2 - \underbrace{\frac{3}{2}x}_{2ab} = 2$$

$$\left(\frac{\left(\frac{3}{2}\right)}{2}\right)^2 = \left(\frac{3}{4}\right)^2$$

(complementary term)

$$\Rightarrow \quad x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 = 2 + \left(\frac{3}{4}\right)^2$$

$$\Leftrightarrow \quad \left(x - \frac{3}{4}\right)^2 = 2 + \left(\frac{3}{4}\right)^2$$

(second binomial form)

$$\Leftrightarrow \quad x_{1,2} - \frac{3}{4} = \pm\sqrt{\left(\frac{3}{2}\right)^2 + 2}$$

$$\Rightarrow \quad x_{1,2} = \frac{3}{4} \pm \sqrt{\left(\frac{3}{2}\right)^2 + 2}$$

(corresponds to p/q formula)

$$= \frac{3}{4} \pm \sqrt{2.5625}$$

$$\Rightarrow \quad x_1 = \frac{3}{4} + \sqrt{2.5625} = \; 2.3508$$

$$x_2 = \frac{3}{4} - \sqrt{2.5625} = -0.8508$$

$$\mathbb{S}_{\mathsf{S}} = \{-0.8508; 2.3508\}$$

4.3.2 Cubic Equations with One Variable

General form: $a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \qquad a_i \in \mathbb{R}$

Normal form: $x^3 + ax^2 + bx + c = 0 \qquad a, b, c \in \mathbb{R}$

Solving Cubic Equations with One Variable

(1) The first x that leads to the solution of the normal form is obtained by trial and error. This can be de facto simplified by choosing an integer divisor of the absolute term c as divisor.

(2) Polynomial long division
$\Rightarrow$ quadratic equation

(3) p/q formula / completing the square

Example: $y = x^3 - 3x^2 - 25x - 21$

(1) $x_1 = -3$, since
$(-3)^3 - 3 \cdot (-3)^2 - 25 \cdot (-3) - 21 = 0$

(2) $$(x^3 \quad -3x^2 \; -25x \quad -21) \div (x+3) = \underbrace{x^2-6x-7}_{\text{quadratic equation}}$$

$$\begin{array}{llll}
-\underline{(x^3 & +3x^2)} & & \\
 & -6x^2 & -25x & \\
 & -\underline{(-6x^2} & \underline{-18x)} & \\
 & & -7x & -21 \\
 & & -\underline{(-7x} & \underline{-21)} \\
 & & & 0
\end{array}$$

(3) $$x_{2/3} = -\left(-\frac{6}{2}\right) \pm \sqrt{\left(-\frac{6}{2}\right)^2 + 7}$$

$$x_{2/3} = 3 \pm \sqrt{16}$$

$$x_2 = 3+4 = 7$$

$$x_3 = 3-4 = -1$$

$$\mathbb{S}_{\mathsf{S}} = \{-3;\ -1;\ 7\}$$

Solving Cubic Equations with One Variable without Absolute Term

(1) Factorise x to the smallest degree
⇒ First solution: $x = 0$ and quadratic equation

(2) p/q formula / completing the square

Example: $$x^8 + 2x^7 - 8x^6 = 0$$

(1) $$x^6(x^2+2x-8) = 0$$

x^6 is equal to zero when x is zero: $\Rightarrow x_1 = 0$

(2) $x^2+2x-8=0$

$$x_{2/3} = -\frac{2}{2} \pm \sqrt{\left(\frac{2}{2}\right)^2 + 8}$$

$$x_{2/3} = -1 \pm \sqrt{9}$$

$$x_2 = -1+3 = 2$$

$$x_3 = -1-3 = -4$$

4.3.3 Biquadratic Equations

General form:	$a_4x^4 + a_2x^2 + a_0 = 0$	$a_i \in \mathbb{R}$
Normal form:	$x^4 + ax^2 + c = 0$	$a, c \in \mathbb{R}$

Solving Biquadratic Equations without Absolute Term

(1) Substitution $z = x^2$
$z^2 + cz + d = 0$ (quadratic equation)
(2) p/q formula / completing the square
(3) Solve for z
(4) Resubstitution ($z \rightarrow x^2$)
(5) Solve for x

Example: $x^4 - x^2 - 6 = 0$ denote $x^2 = z$

(1) $z^2 - z - 6 = 0$

(2) $z_{1/2} = \frac{1}{2} \pm \sqrt{\left(-\frac{1}{2}\right)^2 + 6}$ p/q formula

$z_{1/2} = 0.5 \pm \sqrt{6.25}$

$z_1 = 0.5 + 2.5 = 3$

$z_2 = 0.5 - 2.5 = -2$ denote $z = x^2$

(3) $x^2 = -2 \ \vee \ x^2 = 3 \quad \Leftrightarrow \quad x = \sqrt{-2} \quad \vee \quad x = -\sqrt{-2} \quad \vee$
$\vee \quad x = \sqrt{3} \quad \vee \quad x = -\sqrt{3}$

i.e. (since $x = \sqrt{-2} \ \vee \ x = -\sqrt{-2}$ is not defined):

$\mathbb{S}_S = \{-\sqrt{3}; \sqrt{3}\}$

4.3.4 Equations of the n^{th} Degree

General form of an algebraic equation of the n^{th} degree

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0 \qquad a_1 \in \mathbb{R},\ a_n \neq 0$$

There are no general solutions for general equations of 5^{th} and higher degree.

n^{th} degree polynomials:

An n^{th} degree algebraic equation becomes an n^{th} degree polynomial ($=$ n^{th} degree polynomial function), when:

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0 \qquad a_1 \in \mathbb{R},\ a_n \neq 0, \quad n \geq 5$$

4.3.5 Radical Equations

The variable x appears within the radicand (the term from which the square root is extracted). To eliminate roots, non-equivalent transformations ($=$ exponentiation) are necessary. This results in equations of which solutions do not necessarily have to be solutions of the original equation. Sample offered!

Basic equation

$\sqrt[n]{x} = a \quad \Rightarrow \quad x = a^n$ $\quad a_i \in \mathbb{R}$

$x \in \mathbb{R}$, whereby the whole radicand with even n must not be negative.

$x =$ variable

$\sqrt{x+b} = a \quad \Rightarrow \quad x = a^2 - b \quad x \geq -b,\ a \geq 0$

$$\sqrt{cx} + b = a \Rightarrow \quad x = \frac{(a-b)^2}{c} \quad \operatorname{sgn} x = \operatorname{sgn} c\ ;\ c \neq 0$$

The necessary condition for the domain is that all radicands ≥ 0.

Examples:

(1) $\sqrt{2x-3} - 5 = 0$ $\quad | +5$

$\Leftrightarrow \sqrt{2x-3} = 5$ $\quad | ()^2; +3; \div 2$

$\Rightarrow x = 14$

Test: $\sqrt{2 \cdot 14 - 3} - 5 = 0$ $\quad \mathbb{S}_{\mathrm{S}} = \{14\}$

(2) $\sqrt{x-1}+\sqrt{x+6} = \sqrt{5x-1}$

$$(\sqrt{x-1}+\sqrt{x+6})^2 = (\sqrt{5x-1})^2$$

$$\underbrace{(x-1)+2\sqrt{x-1}\sqrt{x+6}+(x+6)}_{\text{binomial form}} = 5x-1 \qquad \mid -(x-1);\ -(x+6)$$

$$2\sqrt{x-1}\sqrt{x+6} = (5x-1)-(x-1)-(x+6) \quad \mid \div 2$$

$$\sqrt{x-1}\sqrt{x+6} = \frac{3x-6}{2} = 1.5x-3 \qquad \mid ()^2$$

$$(x-1)(x+6) = (1.5x-3)^2 \qquad \mid \text{binomial form}$$

$$x^2+6x-x-6 = 2.25x^2-9x+9 \qquad \mid -2.25x^2;\ +9x;\ -9$$

$$-1.25x^2+14x-15 = 0 \qquad \mid \div(-1.25)$$

$$x^2-11.2x+12 = 0$$

$$x_{\frac{1}{2}} = \frac{11.2}{2} \pm \sqrt{\left(-\frac{11.2}{2}\right)^2-12} \qquad \mid p/q \text{ formula}$$

$$x_{\frac{1}{2}} = 5.6 \pm \sqrt{19.36}$$

$$x_1 = 10;\ x_2 = 1.2$$

Test: $\sqrt{10-1}+\sqrt{10+6}-\sqrt{5\cdot 10-1} = 0$

$$\sqrt{1.2-1}+\sqrt{1.2+6}-\sqrt{5\cdot 1.2-1} \approx 0.894 \neq 0 \qquad \mathbb{S}_{\mathsf{S}} = \{10\}$$

4.4 Transcendental Equations

Every non-algebraic equation is called transcendental.

4.4.1 Exponential Equations

The variable appears in the exponent.

Basic equation $a^x = b$ $a, b \in \mathbb{R}^+; a \neq 1$

$x \in \mathbb{R}$

$x =$ variable

(1) Solution: $\log a^x = \log b$

$\Leftrightarrow \quad x \cdot \log\ a = \log\ b$

$$\Leftrightarrow \quad x = \frac{\log\ b}{\log\ a} = \frac{\lg\ b}{\lg\ a} = \frac{\ln\ b}{\ln\ a}$$

The choice of the base plays no role.

(2) If the bases are the same, then:

$a^x = a^c \quad \Rightarrow \quad x = c$

Examples:

(1) $5^{x+1} = 18$

$\Leftrightarrow$ $\log(5^{x+1}) = \log\ 18$

$\Leftrightarrow$ $(x+1)\cdot\log\ 5 = \log\ 18$

$\Leftrightarrow$ $(x+1) = \dfrac{\log\ 18}{\log\ 5}$

$\Leftrightarrow$ $x = \dfrac{\log\ 18}{\log\ 5} - 1 \approx 0.7959$ The choice of the base plays no role.

$\mathbb{S}_{\mathsf{S}} = \{0.7959\}$

(2) $\sqrt[3]{a^{x-1}} = \sqrt{a^{x+3}}$

$\Leftrightarrow$ $a^{\frac{x-1}{3}} = a^{\frac{x+3}{2}}$

$\Leftrightarrow$ $\dfrac{x-1}{3} = \dfrac{x+3}{2}$

$\Leftrightarrow$ $2(x-1) = 3(x+3)$

$\Leftrightarrow$ $2x-2 = 3x+9$

$\Leftrightarrow$ $x = -11$

$\mathbb{S}_{\mathsf{S}} = \{-11\}$

4.4.2 Logarithmic Equations

The argument is presented in logarithmic form.

Basic equation $\log_a x = b \qquad a, x \in \mathbb{R}^+$
$x =$ variable (the argument)

Solution: $x = a^b$

The solution is **not** equivalent regarding the (original) domain of definition.

If the base of the logarithm is the same, then:

$\log_a x = \log_a c \qquad \Rightarrow x = c$

Examples:

(1) $\ln(2x-5) = 25$

Domain of definition

$2x - 5 > 0 \quad \Rightarrow \quad D = \left\{ x \mid x > \frac{5}{2} \right\}$

$\ln(2x-5) = 25$ | extend with e

$\Leftrightarrow \quad e^{\ln(2x-5)} = e^{25}$

$\Leftrightarrow \quad 2x - 5 = e^{25}$

$\Leftrightarrow \quad x = \frac{1}{2}(e^{25}+5) \approx 3.6 \cdot 10^{10}$

$\mathbb{S}_\mathsf{S} = \{3.6 \cdot 10^{10}\}$

(2) $\log(x^2+1) = 2\log(x+2)$

Domain of definition

$x^2+1>0 \quad \Rightarrow \quad -\infty < x < \infty$

$\wedge \quad x+2>0 \quad \Rightarrow \quad x>-2$

$\Rightarrow \quad D = \{x \mid -2 < x < \infty\}$

$\ln(x^2+1) = 2\ln(x+2)$

$\Leftrightarrow \quad \ln(x^2+1) = \ln(x+2)^2 \qquad |$ extend with e

$\Leftrightarrow \quad e^{\ln(x^2+1)} = e^{\ln(x+2)^2}$

$\Leftrightarrow \quad x^2+1 = (x+2)^2$

$\Leftrightarrow \quad x^2+1 = x^2+4x+4$

$\Leftrightarrow \quad -3 = 4x$

$\mathbb{S}_{\mathrm{S}} = \left\{-\frac{3}{4}\right\}$

Remark:

The transformation of a logarithmic equation can lead to the fact that the domain of definition is no longer equivalent.

Example:

$x = x$ with $x \in \mathbb{R}$

$\Rightarrow \quad \ln x = \ln x$ with $x \in \mathbb{R}^+$

The domain of definition has changed during the transformation, thus no equivalence is given.

4.5 Approximation Methods

The following iteration methods are used to determine a zero of non-linear equations.

Solution principle:

The zero x_0 of a (in the relevant interval) continuous real function $f = f(x)$ is derived from the solution of the equation $f(x) = 0$.

4.5.1 Regula falsi (Secant Method)

Condition:

$f = f(x)$ is a (in the relevant interval) continuous, real function with a single zero x_0

Principle:

The zero X_0 is between two (start) values X_b and X_a with $f(x_b) \cdot f(x_a) < 0$

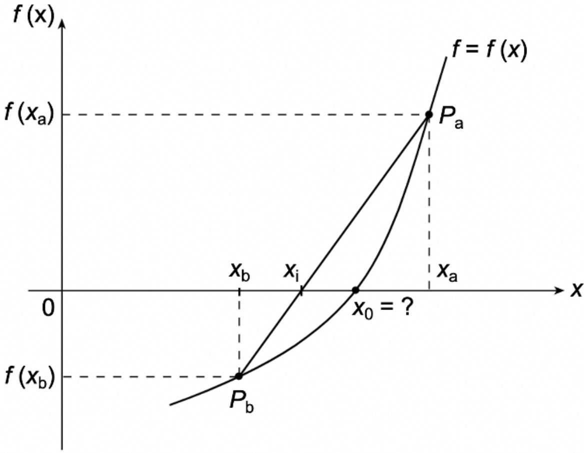


Fig. 4.1: Regula falsi (Secant Method)

The (non-linear) curve is geometrically replaced by the secant between P_b and P_a (secant method, Fig. 4.1). The point x_i, i.e. the intersection of the secant with the x-axis, is calculated as follows:

$$x_i = x_b - f(x_b) \cdot \frac{x_a - x_b}{f(x_a) - f(x_b)}$$

By the (iterative) repetition, the secant gradually gets closer to the zero x_0, so that finally x_0 can be determined (approximately). The **regula falsi** method is numerically stable, i.e. the error decreases or remains the same from iteration to iteration.

Example:

$f(x) = x^3 + 2x - 1$ zero: $x_0 = ?$

First iteration:

Arbitrary choice of two start values with $f(x_{b1}) \cdot f(x_{a1}) < 0$; i.e. one start value is to the left, one to the right of the zero x_0 that is searched for:

$$\begin{aligned} x_{b1} &= 0 \qquad & f(x_{b1}) &= -1 \\ x_{a1} &= 1 \qquad & f(x_{a1}) &= 2 \end{aligned}$$

$$\Rightarrow x_{i1} = 0 + 1 \cdot \frac{1-0}{2+1} = \frac{1}{3} \approx 0.333$$

The iterative repetition of the secant procedure leads to further (successive) approach to x_0:

Second iteration:

around the value $x = 0.333$, e.g.:

$$x_{b2} = 0.2 \qquad f(x_{b2}) \approx -0.592$$

$$x_{a2} = 0.5 \qquad f(x_{a2}) \approx 0.125$$

$$\Rightarrow x_{i2} = 0.2 + 0.592 \cdot \frac{0.5 - 0.2}{0.125 + 0.592} \approx 0.4477$$

Third iteration:

around the value $x = 0.4477$, e.g.:

$$x_{b3} = 0.43 \qquad f(x_{b3}) \approx -0.0605$$

$$x_{a3} = 0.46 \qquad f(x_{a3}) \approx 0.0173$$

$$\Rightarrow x_{i3} = 0.43 + 0.0605 \cdot \frac{0.46 - 0.43}{0.0173 + 0.0605} \approx 0.4533$$

The zero x_0 can be defined more precisely after each iteration. In this example, it is approximately: $x_0 \approx 0.4534$; $f(x_0) \approx 0.0000061453$.

4.5.2 Newton's Method (Tangent Method)

Condition:

The function $f = f(x)$ is differentiable twice in the interval $[x_1; x_2]$. Within this interval, it has a zero x_0 with $f'(x_0) \neq 0$. There is an environment V around x_0 so that the *Newton's method* is applicable and converges for each start value $x_i \in V$ toward x_0.

The procedure fails if the curve of $f = f(x)$ at the respective approximation point is (almost) parallel to the x-axis.

Iteration rule $$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$f'(x_0) \neq 0$ necessary condition,
$|\, f(x) \cdot f''(x) \,| < (f'(x))^2$ sufficient condition.

The (non-linear) curve of the function $f = f(x)$ is geometrically replaced by its tangent at the respective point P_i (tangent method, Fig. 4.2). The starting value x_1 and thus also P_1 can be arbitrarily selected within the interval $[x_1; x_2]$.

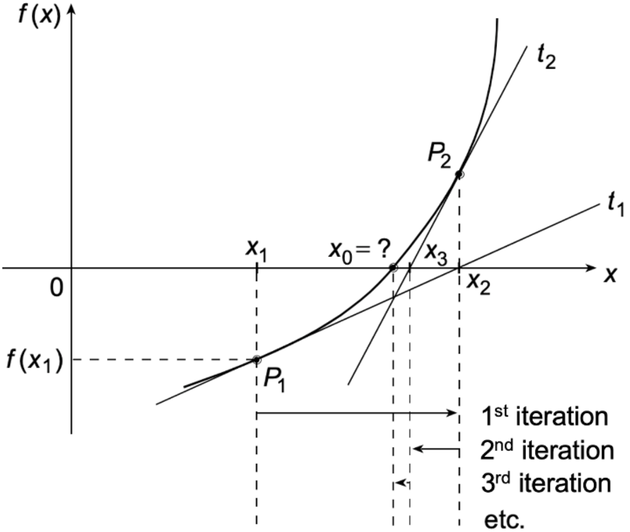


Fig. 4.2: Newton's Method (Tangent Method)

The respective x_{i+1} value is determined by the (aforementioned) respective tangent of f at point P_i with the intersection of the abscissa. x_{i+1} then forms the start value of the subsequent iteration.

By the (iterative) repetition, the tangent gradually gets closer to the zero x_0, so that finally x_0 can be determined (approximately). The *Newton's method* is numerically stable.

Order of convergence $\rho = 2$

Example:

$f(x) = x^3 + 2x - 1$

Zero $x_0 =$?

$f'(x) = 3x^2 + 2 \qquad f''(x_1) = 6x$

First iteration:

Arbitrary selection of a start value:

$$x_1 = 1$$

$$\Rightarrow \qquad f(x_1) = 2; \quad f'(x_1) = 5; \quad f''(x_1) = 6x$$

sufficient condition:

$$| f(x_1) \cdot f''(x) | < (f'(x_1))^2$$

$$= \qquad | 2 \cdot 6 | < 25 \qquad \text{o.k.}$$

$$\Rightarrow \qquad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{2}{5} = \frac{3}{5}$$

The iterative repetition of the secant procedure leads to further (successive) approach to x_0:

Second iteration:

$$x_2 = \frac{3}{5}$$

$$\Rightarrow \qquad f(x_2) = 0.416; \quad f'(x_2) = 3.08; \quad f''(x_2) = 3.6$$

sufficient condition:

$$|\, f(x_2) \cdot f''(x_2) \,| < (f'(x_2))^2$$

$$= \qquad |\, 0.416 \cdot 3.6 \,| < 9.4864 \qquad \text{o.k.}$$

$$\Rightarrow \qquad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{3}{5} - 0.135 = 0.469$$

etc.

The zero x_0 can be defined more precisely after each iteration. In this example, it is approximately: $x_0 \approx 0.4534; f(x_0) \approx 0.0000061453$.

4.5.3 General Approximation Method (Fixed-point Iteration)

Condition:

$f = f(x)$ is a (in the relevant interval) continuous, real function with a single zero x_0. The general approximation method (fixed-point iteration) is shown graphically in Fig. 4.3.

Principle:

$f(x) = 0$ is transformed to $x = g(x)$ (*fixed-point equation*), whereby $g(x)$ is a real, continuous and differentiable function (in the relevant interval).

Iteration rule

$x_{i+1} = g(x_i)$ with $\mid g'(x) \mid < 1$

If $0 \leq g'(x) < 1$, the convergence is monotonous, i.e. one approaches the searched zero x_0 permanently from the same side.

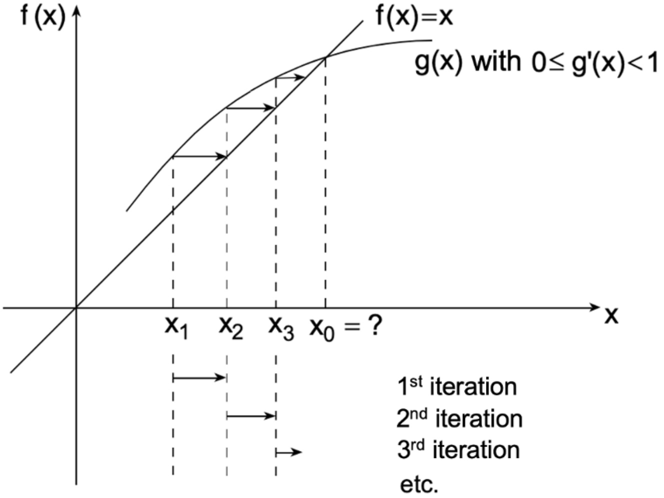


Fig. 4.3: General Approximation Method (Fixed-point Iteration)

If $-1 < g'(x) < 0$, the convergence is oscillating, i.e. two successive approximate values lie on different sides of the zero x_0.

The procedure fails if $\mid g'(x) \mid > 1$ because the inclination angle of the tangent with respect to the curve of g is not between $0°$ to $45°$ resp. $135°$ to $180°$. As a result, the ("approximation") values will successively deviate from x_0. The procedure diverges. In this case, $f(x) = 0$ must be resolved to another term with x (see Example 2).

Example 1:

$$f(x) = x^3 + 2x - 1$$

Zero $x_0 = $?

$$\Leftrightarrow \qquad x = \frac{1 - x^3}{2} = g(x)$$

$$g'(x) = -\frac{3x^2}{2}$$

First iteration:

arbitrary selection of a start value:

$x_1 = 1$

$$g'(x_0) = \frac{3 \cdot 1^2}{2} = -\frac{3}{2}$$

$$\mid g'(x_0) \mid = \frac{3}{2} > 1$$

$\Rightarrow$ Condition violated; A new start value required.

New first iteration:

arbitrary selection of a start value:

$x_1 = 0.5$

$$g'(x_0) = \frac{3 \cdot 0.5^2}{2} = -0.375$$

$\mid g'(x_0) \mid = 0.375 < 1$ o.k.; monotonically convergent

Second iteration:

$$x_2 = g(x_1) = \frac{1-0.5^3}{2} \approx 0.4375$$

Third iteration:

$$x_3 = g(x_2) = \frac{1-0.4375^3}{2} \approx 0.4581$$

Fourth iteration:

$$x_4 = g(x_3) = \frac{1-0.4581^3}{2} \approx 0.4519$$

etc.

Remark

After the third iteration, sufficient accuracy is already reached at the second decimal place.

Example 2:

$$f(x) = x^3 + 2x - 8$$

Zero $x_0 = ?$

$$\Leftrightarrow \qquad x = \frac{8 - x^3}{2} = g(x)$$

$$g'(x) = -\frac{3x^2}{2}$$

First iteration:

arbitrary selection of a start value:

$$x_1 = 1.5$$
$$g'(x_0) = -\frac{3 \cdot 1.5^2}{2} = -3.375$$

$\mid g'(x_0) \mid = 3.375 > 1$ divergent

$\Rightarrow$ Resolve $f(x) = 0$ with the second term of x:

$$x^3 = 8 - 2x \quad \Leftrightarrow \quad x = \sqrt[3]{8 - 2x} = h(x)$$

$$h'(x) = -\frac{2}{3\sqrt[3]{(8-2x)^2}}$$

$$h'(x_0) = -\frac{2}{3\sqrt[3]{(8-2\cdot 1.5)^2}} = 0.228$$

$\mid h'(x_0) \mid = 0.228 < 1$ o.k.; monotonically convergent

Second iteration:

$x_2 = h(x_1) = \sqrt[3]{8-2\cdot 1.5} \approx 1.710$

Third iteration:

$x_3 = h(x_2) = \sqrt[3]{8-2\cdot 1.710} \approx 1.661$

Fourth iteration:

$x_4 = h(x_3) = \sqrt[3]{8-2\cdot 1.661} \approx 1.673$

Fifth iteration:

$x_5 = h(x_4) = \sqrt[3]{8-2\cdot 1.673} \approx 1.670$

etc.

x converges to 1.670 and is therefore fixed point $=$ zero of

$f(x) = x^3 + 2x - 1$

Chapter 5

Linear Algebra

Linear algebra is used, among other things, in the analysis of complex business and economic systems.

5.1 Fundamental Terms

5.1.1 Matrix

An $m \times n$ -matrix A is a rectengular number scheme of m rows and n columns:

$$\begin{array}{rcl} A = & \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix} & \begin{array}{l} \\ \\ \\ \Leftarrow i^{th} \text{ row} \\ \\ \\ \end{array} \\ & \qquad\qquad \Uparrow j^{th} \text{ column} & \end{array}$$

i = row
j = column

The $a_{ij} \in \mathbb{R}$ are called elements of the matrix A.

The first index $i \ (i = 1, ..., m)$ describes the consecutive number of the row, the second index $j \ (j = 1, ..., n)$ the consecutive number of the column.

F. W. Peren, *Math for Business and Economics*,
https://doi.org/10.1007/978-3-662-68860-1_5

Example:

Production conditions in beer production per 100 gallons beer:

Output Input	Lager	Pale Ale
water [gal]	120	140
hops [lbs]	4	6
malt [lbs]	8	3

production matrix $A = \begin{pmatrix} 120 & 140 \\ 4 & 6 \\ 8 & 3 \end{pmatrix}$ 3×2 -matrix

Element:

The elements a_{ij} $(a_{21} = 4)$ indicate how many units of the factor i $(i = 1,2,3)$ are needed to produce one unit of the output j $(j = 1,2)$ (production coefficient).

5.1.2 Equality/Inequality of Matrices

Two matrices $A_{m \times n}$ and $B_{m \times n}$ of the same order are called equal if all elements of A and B coincide.

$A = B$ if $a_{ij} = b_{ij}$ for all i, j

$A \underset{>}{\overset{<}{}} B$ if $a_{ij} \underset{>}{\overset{<}{}} b_{ij}$ for all i, j

$A \underset{\geq}{\overset{\leq}{}} B$ if $a_{ij} \underset{\geq}{\overset{\leq}{}} b_{ij}$ for all i, j

Example:

$$A = \begin{pmatrix} 5 & 7 \\ 7 & 10 \end{pmatrix} \quad B = \begin{pmatrix} 6 & 7 \\ 9 & 10 \end{pmatrix} \quad C = \begin{pmatrix} 4 & 6 & 0 \\ 8 & 9 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 5 & 7 & 1 \\ 9 & 10 & 8 \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \quad & (A \leq B) \quad && (B \neq C) \quad && (C < D) \\ & (A \neq C) && (B \neq D) && \\ & (A \neq D) && && \end{aligned}$$

5.1.3 Transposed Matrix

If the rows and columns of an $m \times n$-matrix A are swapped, the so-called transposed matrix A' or A^t of A with the order $n \times m$ is formed.

Example:

$$A_{2\times 3} = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{pmatrix} \quad \Rightarrow \quad A'_{3\times 2} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 6 & 5 \end{pmatrix}$$

Remark:

If a matrix is transposed twice, the original matrix is restored. The following applies: $(A')' = A$.

5.1.4 Vector

An $m \times 1$-matrix is called column vector, a $1 \times n$-matrix is called row vector. The elements of the vector are called components.

row vector ($1 \times n$):

$$x = (x_1\, x_2 \cdots x_n)_{1\times n}$$

column vector ($n \times 1$):

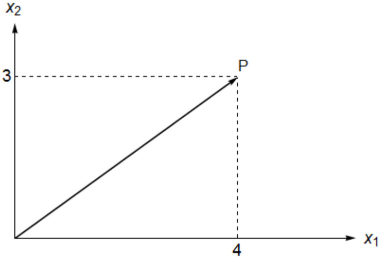
$$x' = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$$

Geometrically, every point P of a k-dimensional space S^k can be described by its k coordinates $x_1, x_2, \cdots, x_k$, which can be summarized as a vector.

Examples:

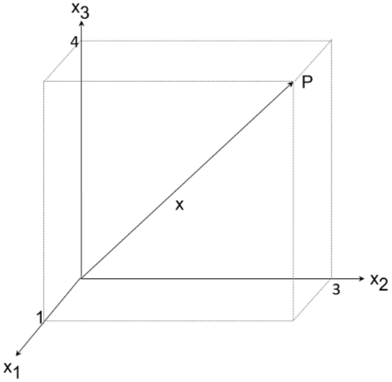
(1) two-dimensional space

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$



(2) three-dimensional space

$$x = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$



Remarks:

- Each matrix $A_{m\times n}$ consists of m row vectors and n column vectors.

  Example: $A = \begin{pmatrix} 1 & 5 & 9 \\ 2 & 7 & 4 \end{pmatrix}$

  row vectors: $(1 \quad 5 \quad 9)$; $(2 \quad 7 \quad 4)$

  column vectors: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}; \begin{pmatrix} 5 \\ 7 \end{pmatrix}; \begin{pmatrix} 9 \\ 4 \end{pmatrix}$

- Each 1×1-matrix is called scalar. Scalars are considered to be real numbers.

  Example:

  The absolute value of a vector $[4] = 4$.

  $$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{or} \qquad x' = (x_1 \quad x_2 \quad \cdots \quad x_n)$$

  The absolute value of a vector is defined as $\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

5.1.5 Special Matrices and Vectors

(1) Zero Matrix

$\rightarrow$ all elements of the matrix = zero

$$0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

(2) Zero Vector

$\rightarrow$ all elements of the vector = zero

(3) Square Matrices

Number of rows $\widehat{=}$ number of columns $\Rightarrow A_{n\times n}$

(4) Main Diagonal of a Matrix

The elements $a_{11}, a_{22}, \cdots, a_{nn}$ form the main diagonal of a square matrix $A_{n\times n}$. The remaining diagonals are called secondary diagonals.

(5) Diagonal Matrix

All elements outside the main diagonals are equal to zero.

$$A_{n\times n} = \begin{pmatrix} \boldsymbol{a_{11}} & 0 & 0 & \cdots & 0 \\ 0 & \boldsymbol{a_{22}} & 0 & \cdots & 0 \\ 0 & 0 & \boldsymbol{a_{33}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \boldsymbol{a_{nn}} \end{pmatrix}$$

with $a_{ij} \neq 0$ for $i = j$

The elements $a_{ij} \neq 0$ form the main diagonal of the matrix $A_{n\times n}$.

(6) Identity Matrix 'I'

All elements of the main diagonal are one, all others are equal to zero.

$$I_{n\times n} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Unit vectors are equivalent to vectors that consist of exactly a single one and otherwise zeros.

<u>Example:</u>

$$i_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad i_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \cdots \quad i_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

(7) Triangular Matrix

All elements (of a square matrix $A_{n\times n}$) on one side of the main diagonal are equal to zero.

<u>Examples:</u>

$$A_{4\times 4} = \begin{pmatrix} 1 & 5 & \sqrt{7} & -1 \\ 0 & 0 & 8 & \pi \\ 0 & 0 & 2 & e \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \widehat{=} \quad \text{upper triangular matrix}$$

$$A_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 4 \end{pmatrix} \quad \widehat{=} \quad \text{lower triangular matrix}$$

(8) Symmetric Matrix

The row elements above the main diagonal are equivalent to the column elements below the main diagonal $\Rightarrow A = A'$.

Example:

$$A = \begin{pmatrix} 3 & \boxed{5} & \boxed{2} \\ \boxed{5} & 7 & \boxed{6} \\ \boxed{2} & \boxed{6} & 9 \end{pmatrix} = A'$$

5.2 Operations with Matrices

5.2.1 Addition of Matrices

Two matrices of the same order are added together by adding the elements in the same position to each other.

$A_{2\times 2}$ + $C_{2\times 3}$ $\Rightarrow$ Addition not possible due to unequal order.

$A_{2\times 2}$ + $B_{2\times 2}$ $\Rightarrow$ Addition possible here because of same order.

$$A_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad B_{m\times n} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$A_{m\times n} + B_{m\times n} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

Examples:

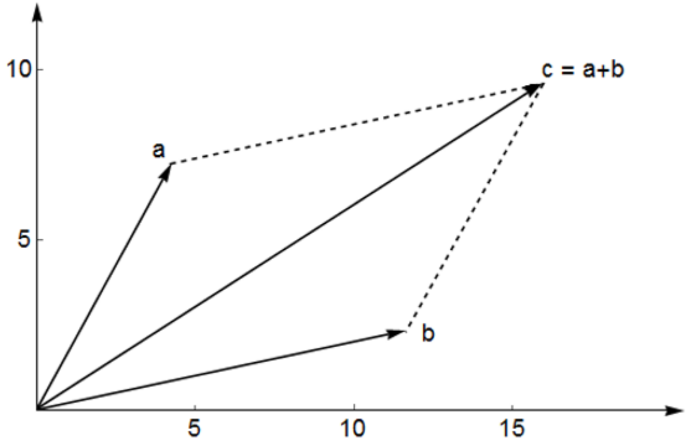
$$A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 2 & 0 \\ 0 & -7 & 1 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1 & 5 & 5 \\ 1 & -3 & 8 \end{pmatrix} \quad A - B = \begin{pmatrix} 3 & 1 & 5 \\ 1 & 11 & 6 \end{pmatrix}$$

Graphical illustration of the addition of two vectors in the two-dimensional space S^2:

$$a = \begin{pmatrix} 5 \\ 6 \end{pmatrix}_{2\times 1} \qquad b = \begin{pmatrix} 11 & 2 \end{pmatrix}_{1\times 2}$$

$$c = a + b' = \begin{pmatrix} 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 11 \\ 2 \end{pmatrix} = \begin{pmatrix} 16 \\ 8 \end{pmatrix}$$



Laws of Addition of Matrices

The following laws apply to matrices of the same order:

(1) $A+B=B+A$ (commutative law)

(2) $A+B+C=(A+B)+C=A+(B+C)$ (associative law)

(3) $A\pm 0=A$

(4) if $A+B=0 \quad \Rightarrow \quad B=-A$ with $A=[a_{ij}]_{m\times n}$

(5) $(A+B)'=A'+B'$

5.2.2 Multiplication of Matrices

5.2.2.1 Multiplication of a Matrix with a Scalar

If there is $k=[k]_{1\times 1}$ with $k\in\mathbb{R}$

and $A=[a_{ij}]_{m\times n}$ with $a_{ij}\in\mathbb{R};\ i=1,\dots,m;\ j=1,\dots,n$

then:

$$k\cdot A_{m\times n}=k\cdot\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn}\end{pmatrix}_{m\times n} = \begin{pmatrix} k\cdot a_{11} & \cdots & k\cdot a_{1n} \\ \vdots & & \vdots \\ k\cdot a_{m1} & \cdots & k\cdot a_{mn}\end{pmatrix}$$

<u>Examples:</u>

(1) $$2\cdot\begin{pmatrix}5\\4\\6\end{pmatrix}=\begin{pmatrix}10\\8\\12\end{pmatrix}$$

(2) $77 \cdot I_{3\times 3} = 77 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 77 & 0 & 0 \\ 0 & 77 & 0 \\ 0 & 0 & 77 \end{pmatrix}$

(3) $\begin{pmatrix} -\frac{9}{11} & \frac{7}{11} & \frac{3}{11} \\ \frac{1}{11} & -\frac{8}{11} & \frac{5}{11} \end{pmatrix} = \frac{1}{11} \cdot \begin{pmatrix} -9 & 7 & 3 \\ 1 & -8 & 5 \end{pmatrix}$

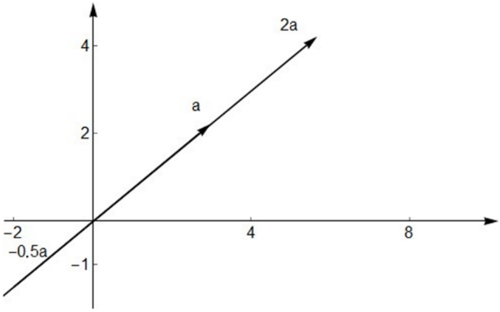
Graphical illustration of the multiplication of a vector a by a scalar k, $(k \in \mathbb{R})$:

$$a = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$2 \cdot a = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \quad ; \quad -0.5 \cdot a = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$k > 1 \quad \widehat{=} \quad$ dilation

$k < 1 \quad \widehat{=} \quad$ compression



Laws of Calculation

$A_{m\times n}$ and $B_{m\times n}$ being matrices of the same order and k,t two real constants (= scalars), the following applies:

(1) $k\cdot(t\cdot A)=(k\cdot t)\cdot A$ (associative law)

(2) $k\cdot A=A\cdot k$ (commutative law)

(3a) $k\cdot(A\pm B)=k\cdot A\pm k\cdot B$ (distributive law)

(3b) $(k\pm t)\cdot A=k\cdot A\pm t\cdot A$ (distributive law)

5.2.2.2 The Scalar Product of Two Vectors

If two vectors a and b of the <u>same order</u> are multiplied by

$$a=\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}_{n\times 1} \qquad b=\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}_{n\times 1}$$

the following is valid:

$$\begin{aligned}a'\cdot b=(a_1\cdots a_n)_{1\times n}\cdot\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}_{n\times 1} &= (a_1\cdot b_1+a_2\cdot b_2+\cdots+a_n\cdot b_n)_{1\times 1}\\ &= \sum_{i=1}^{n}a_i\cdot b_i \qquad \text{with}\quad i=1,...,n\end{aligned}$$

The result of this calculation is equivalent to a real number (= a scalar).

Remark:

row vector · column vector $=$ scalar

$$\underbrace{1 \times \underbrace{n \qquad\qquad n}_{} \times 1}_{} \quad = \quad 1 \times 1$$

Examples:

$$(1) \quad a = \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix}_{3\times 1} \qquad b = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}_{3\times 1}$$

$$\Rightarrow a' \cdot b = (5 \quad 7 \quad 10)_{1\times 3} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}_{3\times 1} = 5 \cdot 2 + 7 \cdot (-1) + 10 \cdot (-2) = -17_{1\times 1}$$

$$(2) \quad a = \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix}_{3\times 1} \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}_{3\times 1}$$

$$\Rightarrow a' \cdot b = (5 \quad 7 \quad 10)_{1\times 3} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}_{3\times 1} = 5 \cdot 1 + 7 \cdot 1 + 10 \cdot 1 = 22_{1\times 1}$$

5.2.2.3 Multiplication of a Matrix by a Column Vector

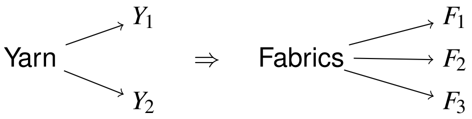
An $(\mathbf{m} \times n)$-matrix A is multiplied by an $(n \times \mathbf{1})$-column vector b by successively multiplying each row vector of a matrix A by the column vector b:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}_{n\times 1} =$$

$$= \begin{pmatrix} a_{11}\cdot b_1 & a_{12}\cdot b_2 & \cdots & a_{1n}\cdot b_n \\ \vdots & \vdots & & \vdots \\ a_{m1}\cdot b_1 & a_{m2}\cdot b_2 & \cdots & a_{mn}\cdot b_n \end{pmatrix}_{m\times 1}$$

The result is an $(\mathbf{m} \times \mathbf{1})$-column vector $\widehat{=}$ a system of linear equations.

Example:



Yarn	Fabrics		
	F_1	F_2	F_3
Y_1	40	100	60
Y_2	80	50	70

} production coefficients

e.g. $40 = 40\ oz\ Y_1$ per $1\ yd\ F_1$
$50 = 50\ oz\ Y_2$ per $1\ yd\ F_2$

Task:

How much yarn, measured in ounces (oz), is needed to produce 50 yards (yd) of F_1, $100\,yd$ of F_2 and $120\,yd$ of F_3?

Solution of this system of linear equations using matrices:

$$A = \begin{pmatrix} 40 & 100 & 60 \\ 80 & 50 & 70 \end{pmatrix}_{2\times 3}$$ matrix of production coefficients

$$b = \begin{pmatrix} 50 \\ 100 \\ 120 \end{pmatrix}_{3\times 1}$$ matrix of production volumes

$$A \cdot b = \begin{pmatrix} 40 \cdot 50 + 100 \cdot 100 + 60 \cdot 120 \\ 80 \cdot 50 + 50 \cdot 100 + 70 \cdot 120 \end{pmatrix}_{2\times 3} = \begin{pmatrix} 19,200 \\ 17,400 \end{pmatrix}_{2\times 1}$$

19,200 oz of yarn Y_1 and **17,400 oz** of yarn Y_2 are required.

General solution: $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mathrel{\widehat{=}}$ a system of linear equations:

$$\Rightarrow A \cdot x = \begin{pmatrix} 40 \cdot x_1 + 100 \cdot x_2 + 60 \cdot x_3 \\ 80 \cdot x_1 + 50 \cdot x_2 + 70 \cdot x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

with $y_1 = 40x_1 + 100x_2 + 60x_3$

$y_2 = 80x_1 + 50x_2 + 70x_3$

5.2.2.4 Multiplication of a Row Vector by a Matrix

A ($\mathbf{1} \times m$) -row vector a is multiplied by an ($n \times \mathbf{m}$) -matrix B by successively multiplying the row vector a by each column vector of matrix B:

$$(a_1\ a_2\ a_3\ \dots\ a_n)_{1\times n} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{pmatrix}_{n\times m} =$$

$$= (a_1 \cdot b_{11} + a_2 \cdot b_{21} + \cdots + a_n \cdot b_{n1} \quad a_1 \cdot b_{12} + a_2 \cdot b_{22} + \cdots + a_n \cdot b_{n2} \quad \cdots \quad a_1 \cdot b_{1m} + a_2 \cdot b_{2m} + \cdots + a_n \cdot b_{nm})_{1\times m}$$

<u>Example:</u>

$$a = (1 \quad 4 \quad 2)_{1\times 3} \qquad B = \begin{pmatrix} 7 & 8 & -2 & 0 \\ 5 & 1 & 0 & 7 \\ 3 & -1 & 5 & -2 \end{pmatrix}_{3\times 4}$$

$$a \cdot B =$$

$$(1 \cdot 7 + 4 \cdot 5 + 2 \cdot 3 \quad 1 \cdot 8 + 4 \cdot 1 + 2 \cdot (-1) \quad 1 \cdot (-2) + 4 \cdot 0 + 2 \cdot 5 \quad 1 \cdot 0 + 4 \cdot 7 + 2 \cdot (-2))_{1\times 4}$$

$$= (33 \quad 10 \quad 8 \quad 24)_{1\times 4}$$

5.2.2.5 Multiplication of Two Matrices

The product of an (**m** $\times p$) -matrix A and a ($p \times$ **n**) -matrix B results in the $(m \times n)$-matrix C, whose elements c_{ij} are each the scalar product of the i^{th} row vector of A and the j^{th} column vector of B:

$$\underset{A}{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mp} \end{pmatrix}_{m\times p}} \quad \cdot \quad \underset{B}{\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pn} \end{pmatrix}_{p\times n}} \quad =$$

$$c_{11} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + \dots + a_{1p} \cdot b_{p1} = \sum_{i=1}^{p} a_{1i} \cdot b_{i1}$$

$$c_{22} = a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + \dots + a_{2p} \cdot b_{p2} = \sum_{i=1}^{p} a_{2i} \cdot b_{i2}$$

$$= \begin{pmatrix} \sum_{i=1}^{p} a_{1i} \cdot b_{i1} & \sum_{i=1}^{p} a_{1i} \cdot b_{i2} & \cdots & \sum_{i=1}^{p} a_{1i} \cdot b_{in} \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^{p} a_{mi} \cdot b_{i1} & \sum_{i=1}^{p} a_{mi} \cdot b_{i2} & \cdots & \sum_{i=1}^{p} a_{mi} \cdot b_{in} \end{pmatrix}_{n\times m}$$

Remark:

Precondition for the multiplication of matrices is that the number of columns (vectors) of matrix A (= 1st factor) matches the number of rows (vectors) of B (= 2nd factor).

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}_{3\times 2} \qquad B = \begin{pmatrix} 1 & -2 & 5 & -7 \\ -3 & 4 & -6 & 8 \end{pmatrix}_{2\times 4}$$

$$A \cdot B = \begin{pmatrix} 1\cdot 1+2\cdot(-3) & 1\cdot(-2)+2\cdot 4 & 1\cdot 5+2\cdot(-6) & 1\cdot(-7)+2\cdot 8 \\ 3\cdot 1+4\cdot(-3) & 3\cdot(-2)+4\cdot 4 & 3\cdot 5+4\cdot(-6) & 3\cdot(-7)+4\cdot 8 \\ 5\cdot 1+6\cdot(-3) & 5\cdot(-2)+6\cdot 4 & 5\cdot 5+6\cdot(-6) & 5\cdot(-7)+6\cdot 8 \end{pmatrix}_{3\times 4} =$$

$$= \begin{pmatrix} -5 & 6 & -7 & 9 \\ -9 & 10 & -9 & 11 \\ -13 & 14 & -11 & 13 \end{pmatrix}_{3\times 4}$$

Rules of Calculation for the Multiplication of Matrices

(1) $(A \cdot B) \cdot C = A \cdot (B \cdot C) = A \cdot B \cdot C$ associative law

$k \cdot (A \cdot B) = (k \cdot A) \cdot B$ k = scalar, with $k \in \mathbb{R}$

(2) $A \cdot (B + C) = A \cdot B + A \cdot C$ distributive law

$(A + B) \cdot C = A \cdot C + B \cdot C$

(3) $A \cdot I = I \cdot A = A$ I = identity matrix

(4) $A \cdot 0 = A \cdot 0 = 0$ 0 = zero matrix

(5) $(A \cdot B)' = (B' \cdot A')' \neq A' \cdot B'$

<u>Example:</u>

$$A = \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \Rightarrow \quad A' = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix} \quad B' = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 2 \cdot 1 + 5 \cdot 3 & 2 \cdot 2 + 5 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 3 & 3 \cdot 2 + 4 \cdot 4 \end{pmatrix} = \begin{pmatrix} 17 & 24 \\ 15 & 22 \end{pmatrix}$$

$$A' \cdot B' = \begin{pmatrix} 2 \cdot 1 + 3 \cdot 2 & 2 \cdot 3 + 3 \cdot 4 \\ 5 \cdot 1 + 4 \cdot 2 & 5 \cdot 3 + 4 \cdot 4 \end{pmatrix} = \begin{pmatrix} 8 & 18 \\ 13 & 31 \end{pmatrix}$$

$$(B' \cdot A')' = \begin{pmatrix} 1 \cdot 2 + 3 \cdot 5 & 1 \cdot 3 + 3 \cdot 4 \\ 2 \cdot 2 + 4 \cdot 5 & 2 \cdot 3 + 4 \cdot 4 \end{pmatrix}' =$$

$$= \begin{pmatrix} 17 & 15 \\ 24 & 22 \end{pmatrix}' = \begin{pmatrix} 17 & 24 \\ 15 & 22 \end{pmatrix}$$

(6) $A \cdot B \neq B \cdot A$

(7) row vector$_{1 \times n}$ $\cdot$ column vector$_{n \times 1}$ = scalar$_{1 \times 1}$
(so-called scalar product)

(8) column vector$_{n \times 1}$ $\cdot$ row vector$_{1 \times n}$ $=$ matrix$_{n \times n}$

(9) $A_{m \times p} \cdot B_{p \times n} = C_{m \times n}$ is only useful if the number of columns of the 1st matrix matches the number of rows of the 2nd matrix.

(10) matrix$_{m \times n}$ $\cdot$ column vector$_{n \times 1}$ $=$ column vector$_{m \times 1}$

(11) matrix · scalar = matrix whose elements are equivalent to the scalar times the elements of the original matrix.

Example:

$$\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \cdot [2] = \begin{pmatrix} 2 & 6 \\ 4 & 10 \end{pmatrix}$$

(12) The following applies to matrix powers:

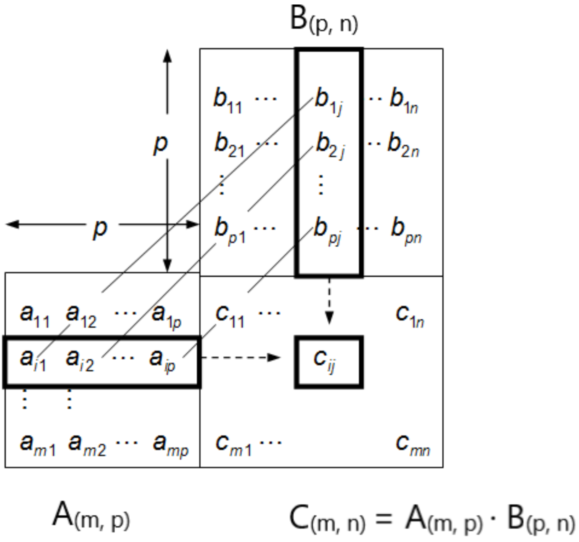
$$A^n = \underbrace{A \cdot A \cdot \ldots \cdot A}_{n\text{-times}}$$

$$A^n \cdot A^m = A^{n+m}$$

$$(A^n)^m = A^{n \cdot m}$$

Falk's Scheme[1]

in general:



At the intersection of the i^{th} row of A and the j^{th} column of B, the scalar product c_{ij} of these columns is shown as the corresponding element of the product matrix $C = A \cdot B$.

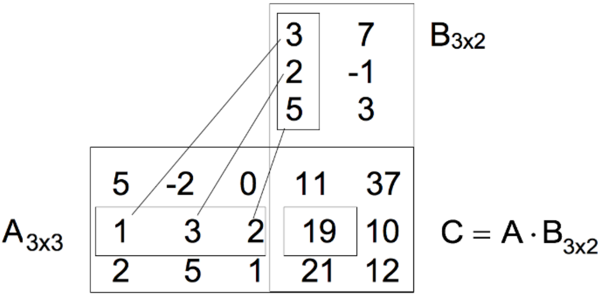
---

[1] Sigurd Falk (1921 - 2016) was a German mathematician.

Example:

$$A = \begin{pmatrix} 5 & -2 & 0 \\ 1 & 3 & 2 \\ 2 & 5 & 1 \end{pmatrix}_{3\times 3} \qquad B = \begin{pmatrix} 3 & 7 \\ 2 & -1 \\ 5 & 3 \end{pmatrix}_{3\times 2} \qquad C = \begin{pmatrix} 11 & 37 \\ 19 & 10 \\ 21 & 12 \end{pmatrix}_{3\times 2}$$

$A_{m\times p} \cdot B_{p\times n} = C_{m\times n}$



5.3 The Inverse of a Matrix

5.3.1 Introduction

Division is not defined for matrices. A matrix equation $A \cdot x = B$ can therefore not easily be “solved” for x. Solution is achieved by forming the so-called inverse matrix.

Excursus: Inverse of a real number:

For real numbers, the inverse is definied as the reciprocal of a number:

The inverse of a number a, $a \in \mathbb{R}\backslash\{0\}$, is equal to $a^{-1} = \dfrac{1}{a}$

$$\Rightarrow \qquad a \cdot x = b \qquad\qquad |\cdot a^{-1} \qquad\qquad \text{[Note: } a^{-1} = \tfrac{1}{a}\text{]}$$

$$\Longleftrightarrow \qquad a^{-1} \cdot a \cdot x = a^{-1} \cdot b$$

$$\iff \quad x = a^{-1} \cdot b$$

$$\iff \quad x = \tfrac{1}{a} \cdot b = \tfrac{b}{a}$$

Inverse of a Matrix:

If for a square matrix $A_{n\times n}$ there exists a matrix $B_{n\times n}$, whose product results in the identity matrix $I_{n\times n}$, then B is called inverse matrix to matrix A. The inverse to A is called A^{-1}.

Example:

$A^{-1} = \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$ is the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, since

$$\begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} A \cdot x &= B \\ \Leftrightarrow \quad A^{-1} \cdot A \cdot x &= A^{-1} \cdot B \qquad \text{[Note: } A^{-1} \cdot A = I\text{]} \\ \Leftrightarrow \quad I \cdot x &= A^{-1} \cdot B \end{aligned}$$

Remark:

Not every square matrix has an inverse.

Example:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad A \cdot A^{-1} = I$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{array}{lcl} 1\cdot b_{11}+0\cdot b_{21}=1 & \Leftrightarrow & b_{11}=1 \\ 1\cdot b_{12}+0\cdot b_{22}=1 & \Leftrightarrow & b_{12}=0 \\ 1\cdot b_{11}+0\cdot b_{21}=1 & \Leftrightarrow & b_{11}=0 \\ 1\cdot b_{12}+0\cdot b_{22}=1 & \Leftrightarrow & b_{12}=1 \end{array}$$

if A^{-1} exists $\Rightarrow A$ regular

if A^{-1} does not exist $\Rightarrow A$ singular

5.3.2 Determination of the Inverse with the Usage of the Gaussian Elimination Method

$$A^{-1}\cdot A = A\cdot A^{-1} = I$$

$$(A/I) \quad \xrightarrow{\text{elementary row operations}} \quad (I/A^{-1})$$

Elementary row operations are:

1. multiplication of a row by a real number $\neq 0$,
2. addition of a row (multiplied by a real number) to another row,
3. interchanging of two rows.

Example:

(1) $\quad A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \qquad A^{-1} = ?$

Extension by I:

$$(A/I) = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

Multiplication of the 1st row by $\frac{1}{3}$:

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \quad | \cdot \frac{1}{3}$$

1st iteration: $\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$

Multiplication of the 1st row by (-2) in addition to the 2nd row
$\Rightarrow$ new 2nd row:

$$\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \quad | \cdot (-2)$$

$$\begin{array}{lr} \begin{pmatrix} -2 & -\frac{4}{3} & -\frac{2}{3} & 0 \end{pmatrix} & \text{1st row new} \\ + \begin{pmatrix} 2 & 1 & 0 & 1 \end{pmatrix} & \text{2nd row} \\ \hline \begin{pmatrix} 0 & -\frac{1}{3} & -\frac{2}{3} & 1 \end{pmatrix} & \text{new 2nd row} \end{array}$$

Result of the 1st iteration: $\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & -\frac{2}{3} & 1 \end{pmatrix}$

Multiplication of the 2nd row by (-3):

$$\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & -\frac{2}{3} & 1 \end{pmatrix} \quad | \cdot (-3)$$

Multiplication of the 2nd row by $\left(-\frac{2}{3}\right)$ in addition to the 1st row
⇒ new 1st row:

2nd iteration: $$\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ & & & \\ 0 & 1 & 2 & -3 \end{pmatrix} \quad | \cdot \left(-\frac{2}{3}\right)$$

$$\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} \quad \text{1st row}$$

$$+ \begin{pmatrix} 0 & -\frac{2}{3} & -\frac{4}{3} & 2 \end{pmatrix} \quad \text{2nd row new}$$

$$\begin{pmatrix} 1 & 0 & -1 & 2 \end{pmatrix} \quad \text{new 1st row}$$

$$(I/A^{-1}) = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -3 \end{pmatrix}$$

Trial: $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(2) $$A = \begin{pmatrix} 2 & 4 & 0 & 1 \\ -5 & 4 & 1 & -7 \\ 1 & 2 & 3 & 4 \end{pmatrix}_{3\times 4}$$

A is not square → singular → inverse does not exist.

Remark:

1. The considered matrix must be square.
2. Not every (square) matrix has an inverse.

Rules of Calculation for Calculating with the Inverse

Let A, B be regular matrices (inverse can be formed).

Then the following applies:

(1) $(A^{-1})^{-1} = A$

(2) $(A^{-1})' = (A')^{-1}$

(3) $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$

(4) $(c \cdot A)^{-1} = \frac{1}{c} \cdot A^{-1}$ with $c \in \mathbb{R} \backslash \{0\}$

(5) $A^{-1} \cdot A = A \cdot A^{-1} = I$

Remark:

If A is regular, which means that A has an inverse, A^{-1}, then A^{-1} is unique, i.e. there is exactly one inverse.

5.4 The Rank of a Matrix

5.4.1 Definition

The rank of a matrix A, rkA, describes the number of linear row (or column) vectors of A.

Mathematical Theorems

- Number of linear independent row vectors $\widehat{=}$ number of linear independent column vectors.
- The rank of a $(m \times n)$-matrix is less than or equal to the number of its rows or columns: $rkA \leq \min\{m, n\}$.
- $rkA = rkA'$

5.4.2 Determination of the Rank of a Matrix

Conversion of matrix A with the help of elementary row operations into a special step structure whose number of steps determines the rank rkA.

$\rightarrow$ Number of steps = rkA

<u>Example 1:</u>

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 5 & 1 & 2 \\ 6 & 2 & 1 \end{pmatrix}$$

Procedure:

1st step: all elements below the 1st step → zero

2nd step: all elements below the 2nd step → zero

$$\begin{pmatrix} 2 & 4 & 3 \\ 5 & 1 & 2 \\ 6 & 2 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} \boxed{1} & 2 & 1.5 \\ 5 & 1 & 2 \\ 6 & 2 & 1 \end{pmatrix}$$

Transformation, so that the element in the 1st row and in the 1st column with the value of 2 becomes 1. To do this, the entire 1st row is multiplied by 0.5. The new 1st row is $(1 \quad 2 \quad 1.5)$.

a) Multiply new 1st row by (-5)

$$\begin{array}{l} (\ -5 \quad -10 \quad -7.5\ ) \\ +(\ \ 5 \qquad\ 1 \qquad\ \ 2\ \ \ ) \\ \hline (\ \ 0 \quad\ -9 \quad -5.5\ ) \text{ new 2nd row} \end{array}$$

b) Multiply new 1st row by (-6)

$$\begin{array}{l} (\ -6 \quad -12 \quad -9\ ) \\ +(\ \ 6 \qquad\ 2 \qquad\ 1\ \ ) \\ \hline (\ \ 0 \quad -10 \quad -8\ ) \text{ new 3rd row} \end{array}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 1.5 \\ 0 & \boxed{-9} & -5.5 \\ 0 & -10 & -8 \end{pmatrix} \quad | \cdot \left(-\tfrac{1}{9}\right)$$

Transformation, so that the element in the 2nd row and in the 2nd column with the value of -9 becomes 1. To do this, the entire 2nd row is multiplied by $-\frac{1}{9}$. The new 2nd row is $(0 \quad 1 \quad 0.61)$.

$$\Rightarrow \begin{pmatrix} 1 & 2 & 1.5 \\ 0 & 1 & 0.6\overline{1} \\ 0 & \boxed{-10} & -8 \end{pmatrix} \quad | \cdot 10 \text{ (Multiplication of the 2nd row by } 10\text{)}$$

$$\Rightarrow \begin{array}{r} (\,0 \quad 10 \quad 6.\overline{1}\;) \\ +\,(\,0 \quad -10 \quad -8\;) \\ \hline (\,0 \quad 0 \quad -1.\overline{9}\;) \end{array}$$

$$\begin{pmatrix} 1 & 2 & 1.5 \\ 0 & 1 & 0.6\overline{1} \\ 0 & 0 & -1.\overline{9} \end{pmatrix}$$

This results in 3 steps $\rightarrow rkA = 3$. The square matrix 3×3 has the rank 3, which means it is regular. The formation of the inverse is possible.

Example 2:

Multiply the first row by $\frac{1}{2}$:

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 5 & 2 & 14 \\ 16 & 16 & 37 \end{pmatrix} \, | \cdot \tfrac{1}{2}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 1{,}5 \\ 5 & 2 & 14 \\ 16 & 16 & 37 \end{pmatrix}$$

Result of the 1st iteration:

$$\Rightarrow \begin{pmatrix} 1 & 2 & 1.5 \\ 0 & -8 & 6.5 \\ 0 & -16 & 13 \end{pmatrix} \quad | \cdot \left(-\tfrac{1}{8}\right)$$

Result of the 2nd iteration:

$$\Rightarrow \begin{pmatrix} 1 & 2 & 1.5 \\ 0 & 1 & -0.8 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow rkA = 2$$

⇒ Formation of the inverse is not possible since A is a 3×3-matrix, while its rank is 2, e.g. it is smaller than 3.

Remark:

If the square $(n \times n)$-matrix has the rank n, it is regular, i.e. the formation of the inverse is possible; if its rank $< n$, the considered matrix is singular, i.e. the formation of the inverse is not possible.

In other words: If A is an $(n \times n)$-matrix, the inverse A^{-1} exists exactly once if $rkA = n$.

5.5 The Determinant of a Matrix

5.5.1 Definition

The determinant, $det\,A$, represents a real number assigned to a square matrix $A = [a_{ij}]_{n\times n}$ as follows:

$$det\,A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot det\,A_{ij}$$

with $j =$ constant, "i.e. expansion along the j^{th} column"

or

$$det\,A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot det\,A_{ij}$$

with $i =$ constant, "i.e. expansion along the i^{th} row"

A_{ij} is the $((n-1)\times(n-1))$-matrix obtained by removing the i^{th} row and the j^{th} column from A:

$$A_{ij} = \begin{pmatrix} a_{11} & \cdots & \cancel{a_{1j}} & \cdots & a_{1n} \\ \vdots & & \cancel{\vdots} & & \vdots \\ \cancel{a_{i1}} & \cancel{\cdots} & \cancel{a_{ij}} & \cancel{\cdots} & \cancel{a_{in}} \\ \vdots & & \cancel{\vdots} & & \vdots \\ a_{n1} & \cdots & \cancel{a_{nj}} & \cdots & a_{nn} \end{pmatrix}$$

Minor

$det\ A_{ij}$ is equivalent to a sub-determinant, the so-called minor.

Remark:

By definition, the determinant of a matrix A is a sum whose summands have alternating positive and negative signs.

$\Sigma(-1)^{i+j}$

The following scheme is recommended as an aid:

$(-1)^{i+j}$				j		
		1	2	3	4	$\cdots$
	1	+	-	+	-	
i	2	-	+	-	+	
	3	+	-	+	-	
	4	-	+	-	+	
	$\vdots$					

5.5.2 Calculation of Determinants

(a) Determinants of $2 \times 2-$Matrices

The determinant of a (2×2)-matrix A is the difference of the products of the diagonal elements:

$$det \begin{pmatrix} a_{11} & & a_{12} \\ & \swarrow\!\!\!\searrow & \\ a_{21} & & a_{22} \end{pmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

Remark:

$\searrow$ = positive sign $\swarrow$ = negative sign

Examples:

$$det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$$

$$det \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix} = 2 \cdot 5 - 4 \cdot 1 = 6$$

or in accordance to the above-mentioned general rule of calulation:

Expansion along the j^{th} column $\rightarrow j =$ constant

e.g. expansion along the 1st column [2]

$$det\ A = \sum_{i=1}^{2} (-1)^{i+j} a_{i1} \ \cdot det\ A_{i1}$$

$A_{i1} =$ removal of the i^{th} row and the 1st column

$$det\ A = +a_{11} \ \ det \ \ A_{11} - a_{21} \ \ det \ \ A_{21} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

Expansion along the i^{th} row $\rightarrow i =$ constant

e.g. $i = 1$

$$det\ A = \sum_{i=1}^{2} (-1)^{i+j} a_{1j} \ \cdot det\ A_{1j}$$

$$det\ A = +a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

---

[2] It may be expanded along any row or column.

(b) Determinants of $3 \times 3-$Matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

e.g. expansion along the column $j = 1$

$$det\ A = \sum_{i=1}^{3} (-1)^{i+j} a_{i1} \cdot det\ A_{i1} = a_{11}\ \ det\ \ A_{11} - a_{21}\ \ det\ \ A_{21} + a_{31}\ \ det\ \ A_{31}$$

$$= a_{11}\ \ det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{21}\ \ det \begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix} + a_{31}\ \ det \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}$$

$$= a_{11}(a_{22} \cdot a_{33} - a_{23} \cdot a_{32}) - a_{21}(a_{12} \cdot a_{33} - a_{13} \cdot a_{32}) + a_{31}(a_{12} \cdot a_{23} - a_{13} \cdot a_{22})$$

Examples:

(1) $A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 0 & 1 \\ 1 & -2 & 5 \end{pmatrix}$

⇒ e.g. expansion along the 2nd row.[3]

$$det\ A = \sum_{i=1}^{3} (-1)^{2+3} a_{2j} \cdot det\ A_{2j}$$

$\Rightarrow 2 + j =$ odd sign, thus minus

$$= -a_{21}\ \ det\ \ A_{21} + a_{22}\ \ det\ \ A_{22} - a_{23}\ \ det\ \ A_{23}$$

$$= -4\ \ det \begin{pmatrix} 3 & -1 \\ -2 & 5 \end{pmatrix} - 0\ \ det \begin{pmatrix} 2 & -1 \\ 1 & 5 \end{pmatrix} + 1\ \ det \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$$

---

[3] It may be expanded along any row or column.

$$= (-4)\cdot(3\cdot 5-(-1)\cdot(-2)) - 0\cdot(2\cdot 5-(-1)\cdot 1) + 1\cdot(2\cdot(-2)-3\cdot 1)$$

$$= (-4)\cdot 13 \qquad\qquad\qquad -0\cdot 11 \qquad\qquad\qquad +1\cdot(-7)$$
$$= -59$$

(2) $$A = \begin{pmatrix} 3 & 9 & 7 \\ 6 & -1 & 8 \\ 2 & 5 & 2 \end{pmatrix}$$

⇒ e.g. expansion along the 3rd column

$$det\ A = \sum_{i=1}^{3} (-1)^{i+1} a_{i3} \cdot det\ A_{i3}$$

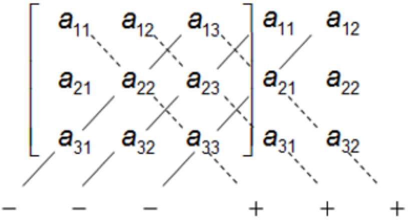
$$= a_{13}\ \ det\ A_{13} - a_{23}\ \ det\ A_{23} + a_{33}\ \ det\ A_{33}$$

$$= 7\ \ det \begin{pmatrix} 6 & -1 \\ 2 & 5 \end{pmatrix} - 8\ \ det \begin{pmatrix} 3 & 9 \\ 2 & 5 \end{pmatrix} + 2\ \ det \begin{pmatrix} 3 & 9 \\ 6 & -1 \end{pmatrix}$$

$$= 7(6\cdot 5+1\cdot 2) - 8(3\cdot 5-9\cdot 2) + 2(3\cdot(-1)-9\cdot 6) = 134$$

<u>Alternative solution:</u> Application of **Sarrus' Rule**[4]
(up to 3×3-matrix)

The first two columns of the original matrix are re-added (on the right), then the sum of the products parallel to the main and parallel to the minor diagonals is formed.



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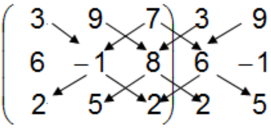
[4] Pierre Frédéric Sarrus (1798 - 1861) was a French mathematician.

Example:

$$= (3\cdot(-1)\cdot 2) + (9\cdot 8\cdot 2) + (7\cdot 6\cdot 5) - (7\cdot(-1)\cdot 2) - (3\cdot 8\cdot 5) - (9\cdot 6\cdot 2) = 134$$



(c) Determinants of ($n \times n$)-Matrices (with n > 3)

Determinants of ($n \times n$)-Triangular Matrices

Analogous to (b) Determinants of 3×3-Matrices, it is also possible to apply the expansion along a row or along a column according to Laplace to (square) matrices of higher order.

$$det\begin{pmatrix} 3 & 4 & 7 & 1 \\ 0 & 2 & 1 & -1 \\ 4 & 5 & -2 & 0 \\ 1 & 2 & 1 & 3 \end{pmatrix} =$$

by means of the expansion e.g. along the 1st row

$$= +3\begin{pmatrix} 2 & 1 & -1 \\ 5 & -2 & 0 \\ 2 & 1 & 3 \end{pmatrix} - 4\begin{pmatrix} 0 & 1 & -1 \\ 4 & -2 & 0 \\ 1 & 1 & 3 \end{pmatrix} + 7\begin{pmatrix} 0 & 2 & -1 \\ 4 & 5 & 0 \\ 1 & 2 & 3 \end{pmatrix} - 1\begin{pmatrix} 0 & 2 & 1 \\ 4 & 5 & -2 \\ 1 & 2 & 1 \end{pmatrix} =$$

$$= +3\cdot(-36) - 4\cdot(-18) + 7\cdot(-27) - 1\cdot(-9) = -216$$

$$det\begin{pmatrix} 3 & 4 & 7 & 1 \\ 0 & 2 & 1 & -1 \\ 4 & 5 & -2 & 0 \\ 1 & 2 & 1 & 3 \end{pmatrix} =$$

by means of the expansion e.g. along the 2nd column

$$= -4\begin{pmatrix} 0 & 1 & -1 \\ 4 & -2 & 0 \\ 1 & 1 & 3 \end{pmatrix} + 2\begin{pmatrix} 3 & 7 & 1 \\ 4 & -2 & 0 \\ 1 & 1 & 3 \end{pmatrix} - 5\begin{pmatrix} 3 & 7 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 3 \end{pmatrix} + 2\begin{pmatrix} 3 & 7 & 1 \\ 0 & 1 & -1 \\ 4 & -2 & 0 \end{pmatrix} =$$

$$= -4(-18) + 2(-96) - 5\cdot 4 + 2\cdot(-38) = -216$$

$$det\begin{pmatrix} 4 & 15 & -27 & -13 \\ 0 & 2 & -8 & 46 \\ 0 & 0 & 5 & 107 \\ 0 & 0 & 0 & -7 \end{pmatrix} =$$

by means of the expansion e.g. along the 3rd row

$$= +0\begin{pmatrix} 15 & -27 & -13 \\ 2 & -8 & 46 \\ 0 & 0 & -7 \end{pmatrix} - 0\begin{pmatrix} 4 & 15 & -13 \\ 0 & 2 & 46 \\ 0 & 0 & -7 \end{pmatrix} - 107\begin{pmatrix} 4 & 15 & -27 \\ 0 & 2 & -8 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= +0\cdot 462 - 0\cdot 224 + 5\cdot(-56) - 107\cdot 0 = -20$$

As an alternative, the determinant of an upper or lower triangular matrix can be formed by the product of the elements of the main diagonals.

$$det\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{22} & a_{23} & \cdots & a_{2n} \\ & & a_{33} & \cdots & a_{3n} \\ & & & & \vdots \\ 0 & & & & a_{nn} \end{pmatrix} = a_{11}\cdot a_{22}\cdot\ldots\cdot a_{nn} = \prod_{i=1}^{n} a_{ii}$$

<u>Example:</u>

$$det\begin{pmatrix} 4 & 15 & -27 & -13 \\ & 2 & -8 & 46 \\ & & 5 & 107 \\ 0 & & & -7 \end{pmatrix} = 4\cdot 2\cdot 5\cdot(-7) = -280$$

Determinants of 4×4-Matrices

Example:

$$A = \begin{pmatrix} 3 & 4 & 7 & 1 \\ 0 & 2 & 1 & -1 \\ 4 & 5 & -2 & 0 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

Formation of an upper triangular matrix

$$\Rightarrow \begin{pmatrix} 3 & 4 & 7 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & -\frac{67}{6} & -\frac{9}{6} \\ 0 & 0 & 0 & 3.2 \end{pmatrix}$$

$$det\, A = 3 \cdot 2 \cdot \left(-\frac{67}{6}\right) \cdot 3.223 = -216$$

$\rightarrow$ simplest method with 4×4-matrices

5.5.3 Characteristics of Determinants

Let A be an $(n \times n)$-matrix, then:

(1) $det\, A = det\, A'$

Example: $det \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = det \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = 3$

(2) Swapping two rows/columns changes the sign of the determinants and thus the result.

Example: $det \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = 3 \qquad det \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} = -3$

(3) The row/column vectors of the matrix A are linearly dependent, if: $det\ A = 0$,

i.e. A is singular → no inverse formation possible.

Example: $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = 0 \rightarrow$ linearly dependent

(4) $detA = 0$, if all elements of a row or a column are zero.

(5) For two $(n \times n)$-matrices, A, B, the following applies:

$det(A \cdot B) = detA \cdot detB$

However, in general:

$det(A + B) \neq detA + detB$

5.6 The Adjoint of a Matrix

5.6.1 Definition

The adjoint of a matrix is the transpose of the cofactor matrix. Multiplying the minor $detA_{ij}$ by the factor $(-1)^{i+j}$ results in the cofactor α_{ij} of the element a_{ij}. If the cofactors α_{ij}, with $i, j = 1, ..., n$, are combined into a matrix, the cofactor matrix $[\alpha_{ij}]_{n \times n}$ is formed. If the cofactor matrix $[\alpha_{ij}]_{n \times n}$ is transposed, the adjoints are finally formed.

$[\alpha_{ij}]'_{n \times n} = A_{ad}$ of the original matrix A.

Example: $A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 3 & 7 & 5 \end{pmatrix}$

$\Rightarrow A_{3\times 3}$ exist $3^2 = 9$ sub-determinants (= minors).

$$detA_{11} = det\begin{pmatrix} 0 & 2 \\ 7 & 5 \end{pmatrix} = 0\cdot 5 - 2\cdot 7 = -14 \Rightarrow \alpha_{11} = (-1)^{1+1}\cdot(-14) = -14$$

$$detA_{12} = det\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = 1\cdot 5 - 2\cdot 3 = -1 \Rightarrow \alpha_{12} = (-1)^{1+2}\cdot(-1) = 1$$

$$detA_{13} = det\begin{pmatrix} 1 & 0 \\ 3 & 7 \end{pmatrix} = 1\cdot 7 - 0\cdot 3 = 7 \Rightarrow \alpha_{13} = (-1)^{1+3}\cdot 7 = 7$$

$$detA_{21} = det\begin{pmatrix} 2 & 4 \\ 7 & 5 \end{pmatrix} = 2\cdot 5 - 4\cdot 7 = -18 \Rightarrow \alpha_{21} = (-1)^{2+1}\cdot(-18) = 18$$

$$detA_{22} = det\begin{pmatrix} 3 & 4 \\ 3 & 5 \end{pmatrix} = 3\cdot 5 - 4\cdot 3 = 3 \Rightarrow \alpha_{22} = (-1)^{2+2}\cdot 3 = 3$$

$$detA_{23} = det\begin{pmatrix} 3 & 2 \\ 3 & 7 \end{pmatrix} = 3\cdot 7 - 2\cdot 3 = 15 \Rightarrow \alpha_{23} = (-1)^{2+3}\cdot 15 = -15$$

$$detA_{31} = det\begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix} = 2\cdot 2 - 4\cdot 0 = 4 \Rightarrow \alpha_{31} = (-1)^{3+1}\cdot 4 = 4$$

$$detA_{32} = det\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = 3\cdot 2 - 4\cdot 1 = 2 \Rightarrow \alpha_{32} = (-1)^{3+2}\cdot 2 = -2$$

$$detA_{33} = det\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} = 3\cdot 0 - 2\cdot 1 = -2 \Rightarrow \alpha_{33} = (-1)^{3+3}\cdot(-2) = -2$$

$$\Rightarrow |\alpha_{ij}|_{3\times 3} = \begin{pmatrix} -14 & 1 & 7 \\ 18 & 3 & -15 \\ 4 & -2 & -2 \end{pmatrix} = \text{cofactor matrix}$$

$$\Rightarrow A_{ad} = |\alpha_{ij}|'_{3\times 3} = \begin{pmatrix} -14 & 18 & 4 \\ 1 & 3 & -2 \\ 7 & -15 & -2 \end{pmatrix} = \text{adjoint}$$

5.6.2 Determination of the Inverse with the Usage of the Adjoint

The following is valid:

$$A^{-1} = \frac{1}{detA} \cdot A_{ad} = \frac{1}{detA} \cdot [\alpha_{ij}]'_{n\times n}$$

Remark:

The matrix A is only regular (invertible), if $det\ A \neq 0$.

Example:

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 3 & 7 & 5 \end{pmatrix}$$

$A^{-1} = ?$

The following is valid:

$$A^{-1} = \frac{1}{detA} \cdot A_{ad}$$

$$[\alpha_{ij}]_{3\times 3} = \begin{pmatrix} -14 & 1 & 7 \\ 18 & 3 & -15 \\ 4 & -2 & -2 \end{pmatrix} = \text{cofactor matrix}$$

$\Rightarrow$ see example in chapter 5.6.1.

$$\Rightarrow A_{ad} = \begin{pmatrix} -14 & 18 & 4 \\ 1 & 3 & -2 \\ 7 & -15 & -2 \end{pmatrix} = \text{adjoint}$$

$detA = ?$

Calculation e.g. with the aid of Sarrus' Rule (see chapter 5.5.2).

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 3 & 7 & 5 \end{pmatrix} \begin{matrix} 3 & 2 \\ 1 & 0 \\ 3 & 7 \end{matrix}$$

$$det\,A = 3\cdot 0\cdot 5 + 2\cdot 2\cdot 3 + 4\cdot 1\cdot 7 - 4\cdot 0\cdot 3 - 3\cdot 2\cdot 7 - 2\cdot 1\cdot 5 = -12$$

$$A^{-1} = \frac{1}{detA}\cdot A_{ad} = -\frac{1}{12}\cdot \begin{pmatrix} -14 & 18 & 4 \\ 1 & 3 & -2 \\ 7 & -15 & -2 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{14}{12} & -\frac{18}{12} & -\frac{4}{12} \\ -\frac{1}{12} & -\frac{3}{12} & \frac{2}{12} \\ -\frac{7}{12} & \frac{15}{12} & \frac{2}{12} \end{pmatrix} = \begin{pmatrix} \frac{7}{6} & -\frac{3}{2} & -\frac{1}{3} \\ -\frac{1}{12} & -\frac{1}{4} & \frac{1}{6} \\ -\frac{7}{12} & \frac{5}{4} & \frac{1}{6} \end{pmatrix}$$

Chapter 6

Combinatorics

6.1 Introduction

A basic task of combinatorics is to determine the number of possible arrangements (permutations) for a (basic) population of N different elements $e_1, e_2, \ldots, e_N$.

<u>Example</u>:

A (basic) population of $N = 3$ elements e_1, e_2, e_3 results in six different arrangements:

$$\left.\begin{array}{l} e_1e_2e_3 \\ e_1e_3e_2 \\ e_2e_1e_3 \\ e_2e_3e_1 \\ e_3e_1e_2 \\ e_3e_2e_1 \end{array}\right\} \Rightarrow 3! = 1 \cdot 2 \cdot 3 = 6$$

In general: for N different elements there are $N!$ arrangements (= so-called permutations).

$N! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (N-1) \cdot N$ ($N!$, read: "N factorial")

Remark: $0! = 1$

Another important task of combinatorics is the determination of possible arrangements when selecting n elements from a basic population of N elements.

F. W. Peren, *Math for Business and Economics*,
https://doi.org/10.1007/978-3-662-68860-1_6

Urn Model:

From an urn with a total of N balls, n balls are drawn.

Without repetition of elements:	in each arrangement, each element occurs once at most; sampling without replacement
With repetition of elements:	at least one element can occur multiple times; sampling with replacement
Order is relevant:	the swapping of elements within an arrangement results in a new arrangement (so-called variation)
Order is irrelevant:	the swapping of elements within an arrangement does not result in a new arrangement (so-called combination)

Examples:

With each $N = 3$ and $n = 2$ elements

(1) possible arrangements **without** repetition of single elements

(a) order is **relevant:**

$$\left.\begin{array}{l} e_1e_2 \\ e_1e_3 \\ e_2e_1 \\ e_2e_3 \\ e_3e_1 \\ e_3e_2 \end{array}\right\} \Rightarrow \frac{N!}{(N-n)!} = \frac{3!}{(3-2)!} = \frac{6}{1} = 6 \text{ variations}$$

(b) order is **irrelevant:**

$$\left.\begin{array}{l} e_1e_2 = e_2e_1 \\ e_1e_3 = e_3e_1 \\ e_2e_3 = e_3e_2 \end{array}\right\} \Rightarrow \binom{N}{n} = \frac{N!}{n!(N-n)!} = \binom{3}{2} = \frac{3!}{2!1!} = \frac{6}{2\cdot 1}$$

$= 3$ combinations

$\binom{N}{n}$ read: "N choose n" (binomial coefficient)

Remark:

$$\binom{N}{0} = \binom{N}{N} = 1$$

$$\binom{N}{1} = \binom{N}{N-1} = N$$

$$\binom{N}{n} = \binom{N}{N-n}$$

(2) possible arrangements **with** repetition of single elements

(a) order is **relevant:**

$$\left.\begin{array}{l} e_1e_1 \\ e_2e_1 \\ e_3e_1 \\ e_1e_2 \\ e_2e_2 \\ e_3e_2 \\ e_1e_3 \\ e_2e_3 \\ e_3e_3 \end{array}\right\} \Rightarrow N^n = 3^2 = 9 \text{ variations}$$

(b) order is **irrelevant:**

$$\left.\begin{array}{l} e_1e_1 \\ e_1e_2 \\ e_1e_3 \\ e_2e_2 \\ e_2e_3 \\ e_3e_3 \end{array}\right\} \Rightarrow \binom{N+n-1}{n} = \binom{3+2-1}{2} = \binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{24}{2\cdot 2}$$

$= 6$ combinations

6.2 Permutations

Definition:

A permutation P of N different elements corresponds to the number of possible arrangements in a **(full) survey** of all elements. Here it can be distinguished whether an element exists only once in the basic population (without repetition) or whether an element occurs several times and is therefore not distinguishable (with repetition).

without repetition	with repetition
Every element occurs exactly once per arrangement. $P_{\text{w/o.rep.}} = N!$	The i^{th} element occurs multiple times, i.e. repeatedly. $P_{\text{w/rep.}} = \frac{N!}{n_1!\cdot n_2!\cdot\ldots\cdot n_k!}$
Example: elements: $e_1, e_2 \Rightarrow N = 2$ $P_{\text{w/o.rep.}} = 2! = 2$ namely $e_1e_2;\ e_2e_1$	Example: elements: e_1, e_2 $\Rightarrow n_{e_1} = 2;\ n_{e_2} = 1$ $P_{\text{w/rep.}} = \frac{(2+1)!}{2!\cdot 1!} = \frac{1\cdot 2\cdot 3}{2\cdot 1\cdot 1} = 3$ namely $e_1e_1e_2;\ e_1e_2e_1;\ e_2e_1e_1$

Examples:

Permutation without Repetition

Four horses compete in a horse race. How many possibilities are there for the horses to reach the finish line in different orders?

Solution: $P_{\text{w/o.rep.}} = N! = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

How many ways can five women and three men pass through a revolving door?

Solution: $P_{\text{w/o.rep.}} = N! = 8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40,320$

There are four German and three English books on a bookshelf. The German books should be placed on the left side of the bookshelf and the English books on the right side of the bookshelf. How many ways are possible to arrange the books?

Solution: $P_{\text{w/o.rep.}} = N = N_1! \cdot N_2! = 4! \cdot 3! = 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 = 144$

Permutation with Repetition

In an urn there are two red and three white balls. How many possibilities are there to put them in order?

Solution: $P_{\text{w/rep.}} = \dfrac{N!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!} = \dfrac{5!}{2! \cdot 3!} = \dfrac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1) \cdot (3 \cdot 2 \cdot 1)} = 10$

How many possibilities are there to arrange the individual letters of the word MISSISSIPPI?

Solution: in total 11 letters $\Rightarrow N! = 11$

1xM, 4xI, 4xS, 2xP $\Rightarrow n_1 = 1; \quad n_2 = 4; \quad n_3 = 4; \quad n_4 = 2$

$$P_{\text{w/rep.}} = \frac{N!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!} = \frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = \frac{39,916,800}{1,152} = 34,650$$

6.3 Variations

Definition:

A variation V of n elements from a basic population of N different elements is equivalent to the number of possible arrangements, if the order of the elements in the arrangement is **relevant**. Here it can be distinguished whether elements occur only once (without repetition) or whether they occur multiple times (with repetition).

without repetition	with repetition
$V_{\text{w/o.rep.}} = \frac{N!}{(N-n)!}$	$V_{\text{w/rep.}} = N^n$

Example: elements: e_1, e_2, e_3 $\Rightarrow N = 3; n = 2$

$V_{\text{w/o.rep.}} = \frac{3!}{(3-2)!} = 6$ namely $e_1e_2,\ e_1e_3,\ e_2e_1,\ e_2e_3,$ e_3e_1, e_3e_2	$V_{\text{w/rep.}} = 3^2 = 9$ namely $e_1e_1, e_2e_1, e_3e_1, e_1e_2,$ $e_2e_2, e_3e_2, e_1e_3, e_2e_3, e_3e_3$

Examples:

Variation without Repetition

Ten cars participate in a car race. How many possibilities are there to fill the first three places, respecting the order?

Solution: $V_{\text{w/o.rep.}} = \dfrac{N!}{(N-n)!} = \dfrac{10!}{(10-3)!} = \dfrac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} =$

$= 720$

Variation with Repetition

A bicycle lock uses a four-digit code consisting of the numbers 0 to 9. How many possibilities are there if the individual digits may occur several times?

Solution: $V_{\text{w/rep.}} = N^n = 10^4 = 10,000$

6.4 Combinations

Definition:

A combination C of n elements from a basic population of N different elements is meant to be the number of possible arrangements, if the order of the elements in the arrangement is **irrelevant**. Here it can be distinguished whether the elements occur only once (without repetition) or whether they can occur multiple times (with repetition).

without repetition	with repetition
$C_{\text{w/o.rep.}} = \binom{N}{n}$ $= \dfrac{N!}{n! \cdot (N-n)!}$	$C_{\text{w/rep.}} = \binom{N+n-1}{n}$ $= \dfrac{(N+n-1)!}{(N-1)! \cdot n!}$

<u>Example:</u> elements: e_1, e_2, e_3 $\Rightarrow N = 3; n = 2$

$C_{\text{w/o.rep.}} = \binom{3}{2} = \dfrac{3!}{2! \cdot (3-2)!} = 3$	$C_{\text{w/rep.}} = \binom{3+2-1}{2} = \binom{4}{2}$ $= \dfrac{4!}{2! \cdot (4-2)!} = 6$
namely $e_1e_2,\ e_1e_3,\ e_2e_3$	namely $e_1e_1,\ e_1e_2,\ e_1e_3,\ e_2e_2,$ $e_2e_3,\ e_3e_3$

<u>Examples:</u>

Combination without Repetition

In a lotto 6 out of 49, exactly six of the 49 numbers should be ticked. How many possibilities are there if the order is not taken into regard and a drawn number may only occur once?

Solution: $C_{\text{w/o.rep.}} = \binom{N}{n} = \binom{49}{6} = \dfrac{49!}{6! \cdot (49-6)!} = \dfrac{49!}{6! \cdot 43!} = 13{,}983{,}816$

For a project, a company wants to put together a team of three employees. How many possibilities are there to form a team when twelve employees are available?

Solution: $C_{\text{w/o.rep.}} = \binom{N}{n} = \binom{12}{3} = \dfrac{12!}{3! \cdot (12-3)!} = 220$

In a lecture hall there are nine lamps that can be switched on and off independently of each other. How many possibilities are there if a minimum of six lamps must be lit?

Solution: minimum of 6 lamps $\Rightarrow$ exactly 6, 7, 8 or 9 lamps are lit

$$N=9; \qquad n_1=6, \quad n_2=7, \quad n_3=8, \quad n_4=9$$

$$C_{\text{w/o.rep.}}=\binom{N}{n}\Rightarrow\binom{9}{6}+\binom{9}{7}+\binom{9}{8}+\binom{9}{9}$$

$$=84+36+9+1=130$$

Combination with Repetition

Example 1:

From an urn with six different coloured balls, four balls are to be drawn and put back (with repetition). How many possibilities are there if the order is disregarded?

Solution: $C_{\text{w/rep.}}=\dfrac{(N+n-1)!}{(N-1)!\cdot n!}=\dfrac{(6+4-1)!}{(6-1)!\cdot 4!}=\dfrac{9!}{5!\cdot 4!}=126$

Sweets Ltd. produces candies in the flavours apple, orange, banana, pineapple and blueberry. How many possible candy mixtures are there, if 15 candies fit into one bag and the candies are filled into the bags randomly?

Solution: $C_{\text{w/rep.}}=\dfrac{(N+n-1)!}{(N-1)!\cdot n!}=\dfrac{(15+5-1)!}{(15-1)!\cdot 5!}=\dfrac{19!}{14!\cdot 5!}=11{,}628$

Example 2:

A jewelery manufacturer produces multicolored pearl necklaces using seven different colors. On one chain 40 pearls are threaded.

How many pearl combinations are possible if the pearls are threaded onto the chain purely at random and the individual colors are allowed to repeat?

Solution: $C_{\text{w/rep..}}$: $N = 7, n = 40,$ i.e. here $N < n$

$$\binom{N+n-1}{n} = \binom{7+40-1}{40} = \binom{46}{40} = 46 \boxed{\text{nCr}} \text{-button } 40 = 9{,}366{,}819$$

There are $9{,}366{,}819$ pearl combinations for this necklace.

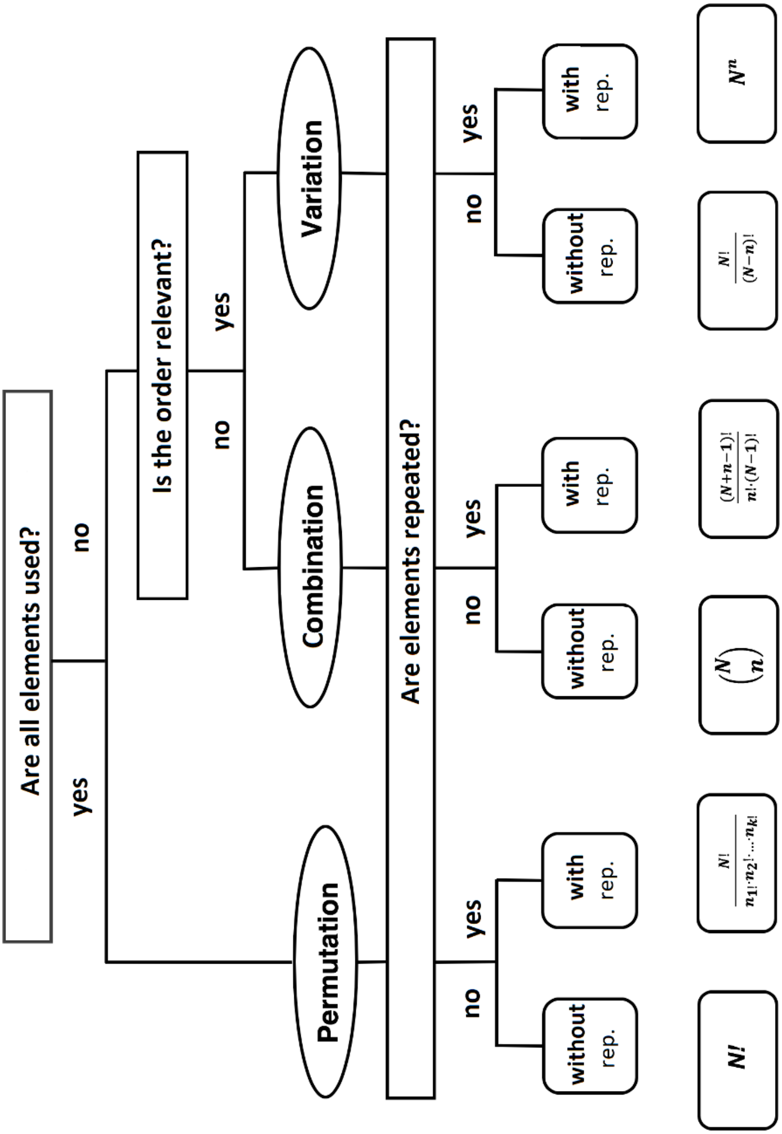
Overview of Basic Combinatorial Formulas:

Selection from a basic population → partial survey $n < N$

repetition / order	**without** rep. of single elements	**with** rep. of single elements
order **relevant** → variation	$\frac{N!}{(N-n)!}$	N^n
order **irrelevant** → combination	$\binom{N}{n}$	$\binom{N+n-1}{n}$

All elements are considered → complete survey $n = N$

	without rep. of single elements	**with** rep. of single elements
permutation	$N!$	$\frac{N!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}$


Are all elements used?
yes
no
Is the order relevant?
no
yes
Permutation
Combination
Variation
Are elements repeated?
no
yes
no
yes
no
yes
without rep.
with rep.
without rep.
with rep.
without rep.
with rep.
$N!$
$\frac{N!}{n_1!\cdot n_2!\cdot\ldots\cdot n_k!}$
$\binom{N}{n}$
$\frac{(N+n-1)!}{n!\cdot(N-1)!}$
$\frac{N!}{(N-n)!}$
N^n

Chapter 7

Financial Mathematics

7.1 Calculation of Interest

7.1.1 Fundamental Terms

Interest z	interest is the charge for a loaned capital. - debit interest is interest that must be paid. - credit interest is interest that is received.
Interest Rate i	determines what percentage of the initial capital is to be paid at the end of an interest period on the initial capital. The interest rates are classified as follows: - according to the length of the interest period: • annual interest rate (year) • interest rate during the year (fraction of a year, e.g. quarter) - according to the calculated reference value: • interest in arrears (initial capital) • advance interest (final capital) Remark: The standard case is an annual interest rate in arrears.

F. W. Peren, *Math for Business and Economics*,
https://doi.org/10.1007/978-3-662-68860-1_7

7.1.2 Annual Interest

7.1.2.1 Simple Interest Calculation

Interest Factor q	$q = 1 + i$ with i = interest rate in decimal notation (e.g. 0.01 for 1%)
Interest Period	the period between two interest payments
Initial Capital C_0	capital at the beginning of the period (also called cash value or present value)
Final Capital C_n	capital after the n^{th} (interest) period (at the end of the period)

n is the period measured in years

t is the period measured in days

The interest must always be calculated from the initial capital C_0, i.e. the annual interest due always remains the same.

Interest $z_{1,2,3,\ldots,n} = C_0 \cdot i$

$z = C_0 \cdot n \cdot i$ (for the period n)

Interest Portion $z = \dfrac{C_0 \cdot i}{12}$ (per month)

$z = C_0 \cdot \dfrac{t \cdot i}{360}$ (for t days)

Final Capital $C_n = C_0 \cdot (1 + n \cdot i)$

$$C_n = C_0 + n \cdot z_{1,\,2,\,3,\,\ldots,\,n} = C_0 + n \cdot C_0 \cdot i$$

Commercial Interest Formula:

$$C_n = C_0 \cdot \left(1 + \frac{t \cdot i}{360}\right) = C_0 + z$$

The equation $C_n = C_0 \cdot (1 + n \cdot i)$ forms the basis for the calculation of C_0, i and n.

By solving it accordingly, the following equations are obtained:

Initial Capital $C_0 = \dfrac{C_n}{1 + n \cdot i}$

Interest Rate $i = \dfrac{1}{n} \cdot \left(\dfrac{C_n}{C_0} - 1\right)$

Period $n = \dfrac{1}{i} \cdot \left(\dfrac{C_n}{C_0} - 1\right)$

<u>Example:</u>

Final Capital $C_0 = \$2{,}000$; $t = 200$ days;
interest rate in percentage $= 10\,\%$

$$z = \$2{,}000 \cdot \left(\frac{200}{360}\right) \cdot 0.1 = \$111.11$$

$$C_n = \$2{,}000 \cdot \left(1 + \frac{200}{360} \cdot 0.1\right) = \$2{,}000\ +\ \$111.11$$

$$C_n = \$2{,}111.11$$

Initial Capital $C_n = \$15{,}000$; $n = 8$ years;

interest rate in percentage $= 5.3\,\%$

$$C_0 = \frac{\$15{,}000}{1+8\cdot 0.053}$$

$$C_0 = \$10{,}533.71$$

Interest Rate $C_0 = \$840$; $C_n = \$1{,}070$; $n = 4$ years

$$\text{interest rate} = \frac{1}{4}\cdot\left(\frac{1{,}070}{840}-1\right)$$

$$\text{interest rate} = 0.0685$$

$$\text{interest rate in percentage} = 6.85\,\%$$

Period $C_0 = \$5{,}000$; $C_n = \$7{,}000$;

interest rate in percentage $= 5\,\%$

$$n = \frac{1}{0.05}\cdot\left(\frac{\$7{,}000}{\$5{,}000}-1\right) = 8 \text{ years}$$

7.1.2.2 Compound Computation of Interest

Interest claims that arise during the period are added to the interest-bearing capital at the end of the year. In the following interest periods, the interest of the previous interest periods is also included.

Final Capital $C_1 = C_0\cdot(1+i)$

$$C_2 = C_0\cdot(1+i)^2$$

$$C_3 = C_0 \cdot (1+i)^3$$

$$\vdots$$

$$C_n = C_0 \cdot (1+i)^n = C_0 \cdot q^n$$

Initial Capital $C_0 = C_n \cdot (1+i)^{-n} = \frac{C_n}{q^n} = C_n \cdot q^{-n}$

Interest Rate $i = \sqrt[n]{\frac{C_n}{C_0}} - 1$

Period $C_n = C_0 \cdot (1+i)^n$

$$\Leftrightarrow \log(1+i)^n = \log(C_n) - \log(C_0)$$

$$\Leftrightarrow n \cdot \log(1+i) = \log(C_n) - \log(C_0)$$

$$\Leftrightarrow n = \frac{\log(C_n) - \log(C_0)}{\log(1+i)} = \frac{\log(C_n) - \log(C_0)}{\log(q)}$$

<u>Example:</u>

Final Capital $C_0 = \$12,500$; $n = 6$ years;
interest rate in percentage $= 4\,\%$

$$C_6 = \$12,500 \cdot (1+0.04)^6 = \$12,500 \cdot 1.04^6$$

$$C_6 = \$15,816.49$$

Initial Capital $C_n = \$2,500$; $n = 7$ years;

interest rate in percentage $= 5\,\%$

$$C_0 = \frac{\$2,500}{(1+0.05)^7} = \$2,500 \cdot 1.05^7$$

$$C_0 = \$1,776.70$$

Interest Rate $C_0 = \$2,000$; $C_n = \$4,000$; $n = 8$ years

$$i = \sqrt[8]{\frac{\$4,000}{\$2,000}} - 1 = 0.091$$

Period $C_0 = \$9,050$; $C_n = \$11,000$;
interest rate in percentage $= 3\,\%$

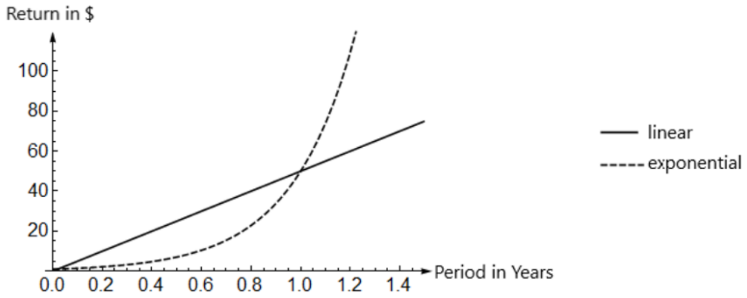
$$\frac{\log(\$11,000) - \log(\$9,050)}{\log(1+0.03)} = \frac{0.08474}{0.01284}$$

$n = 6.6$ years

7.1.2.3 Composite Interest

The composite interest calculation is important for fractional periods (e.g. 1 year + 25 days).

It represents an addition of the simple interest calculation and the exponential interest calculation. If the interest period covers one year, exponential interest is calculated for full years, since exponential interest is more profitable than linear interest from a full interest period onwards. If the time frame includes less than one interest period (during the year), interest is calculated simply since the linear interest calculation generates more revenue than the exponential interest calculation if interest is only calculated for part of a full period.



The graph of the linear versus exponential interest rate shows that the straight line is above the exponential function until the first full interest period is reached, since the exponential function has a flatter slope at the beginning. They intersect at $t = 1$ period (here $t = 1$ year). From this point on, the exponential interest rate generates a higher return than the linear interest rate.

$n = n_1 + n_2$ with

n	total period in years with annual interest periods
n_1	period of a first partial period within the entire interest period n, measured in complete (full) years
n_2	period of a second partial period (remaining period) within the entire interest period n, which corresponds to the remaining (less than one year) period $(n - n_1)$

Example:

Final Capital $C_0 = \$5,000$;
interest rate in percentage $= 3.9\,\%$ p.a.;

from 12.02.2003 to 20.08.2010

February in the years 2004 and 2008 includes 29 days.

a) 30E/360 ISDA (German interest rate method)

The German interest method stipulates that each month is calculated with 30 interest days and a full year with 360 interest days. This means that months that lie entirely between the starting and the ending date of the interest payment period are each counted as 30 days, regardless of the actual number of days they have. If a month has 31 days, the 31st calender day is not an interest day. If the start or the end of the period falls on the 31st of a month, it is treated as the 30th calender day. If the transaction ends on the 28th of February or, in a leap year, on the 29th of February, interest is only calcualted up to this date. If, on the other hand, the transaction goes beyond February, February is treated like every month with 30 days. In this example the 20th of August is not calculated, since according to the commercial interest method the last day of savings deposits on the German capital market does not bear interest (§§ 187, 188 BGB). The first day of investment earns interest, the last day of investment does not.

10.01.2001 to 10.03.2001 → 21 + 30 + 9 = 60 days
28.02.2001 to 10.03.2001 → 3 + 9 = 12 days
10.01.2001 to 28.02.2001 (not a leap year) → 21 + 27 = 48 days
10.01.2000 to 29.02.2000 (leap year) → 21 + 28 = 49 days
10.01.2000 to 01.03.2000 (leap year) → 21 + 30 = 51 days

Months are always counted as 30 days, regardless of the actual number of days. The 31st day of a month is not taken into account. If the end or the beginning of the interest period falls on the 31st day of a month, this is not taken into account. The year always has 12 x 30 = 360 days.

10.01.2001 to 31.03.2001 → 21 + 30 + 29 = 80 days
31.01.2001 to 31.03.2001 → 1 + 30 + 29 = 60 days
30.01.2001 to 31.03.2001 → 1 + 30 + 29 = 60 days

$$C_n = C_0 \cdot \left(1 + \frac{\Delta t_1}{360} \cdot i\right) \cdot (1+i)^n \cdot \left(1 + \frac{\Delta t_2}{360} \cdot i\right)$$

C_n	Final Capital
C_0	Initial Capital
Δt_1	Period from capital contribution to 1st interest payment
Δt_2	Period from last interest payment to end of capital investment
i	Interest Rate p.a.
n	Period measured in years

$$C_n = \$5{,}000 \cdot \left(1 + \frac{319}{360} \cdot 0.039\right) \cdot (1+0.039)^6 \cdot \cdot \left(1 + \frac{229}{360} \cdot 0.039\right)$$

$$C_n = \$6{,}669.00$$

Day of deposit → End of year	Period n-years	End of year → Day of payout
$\underbrace{\text{Remaining period 1}}_{\text{simple interest}}$	$\underbrace{\text{Interest period}}_{\substack{\text{compound computation}\\ \text{of interest}}}$	$\underbrace{\text{Remaining period 2}}_{\text{simple interest}}$
$\frac{30\cdot 10}{360}+\frac{19}{360}$ $=\frac{319}{360}=n_{21}$ 10 months + 19 days Feb. 12th included	6 years $= n_1$	$\frac{30\cdot 7}{360}+\frac{19}{360}$ $=\frac{229}{360}=n_{22}$ 7 months + 19 days Aug. 20th not included

b) 30E/360 ICMA (U.S. interest rate method)

The method is similar to the German commercial interest method, as the interest months are set at 30 days and the interest year at 360 days. The exception is February, which is set to the exact calendar date of 28 or 29 days, provided that the start or end of the period falls on these days. The base year, like the interest month and interest year, is set at 360 days regardless of the number of actual days. The first day of investment does not earn interest, the last day of investment does.

10.01.2001 to 10.03.2001 → 20 + 30 + 10 = 60 days
28.02.2001 (not a leap year) to 10.03.2001 → 0 + 10 = 10 days
10.01.2001 to 28.02.2001 (not a leap year) → 20 + 28 = 48 days
10.01.2000 to 29.02.2000 (leap year) → 20 + 29 = 49 days
10.01.2000 to 01.03.2000 → 20 + 30 + 1 = 51 days

The 31st calendar day of the month counts, provided that the investment ends on this day and the interest period does not begin on the 30th or 31st of another month.

10.01.2001 to 31.03.2001 → 20 + 30 + 31 = 81 days
31.01.2001 to 31.03.2001 → 0 + 30 + 30 = 60 days
30.01.2001 to 31.03.2001 → 0 + 30 + 30 = 60 days

$$C_n = C_0 \cdot \left(1 + \frac{\Delta t_1}{360} \cdot i\right) \cdot (1+i)^n \cdot \left(1 + \frac{\Delta t_2}{360} \cdot i\right)$$

For explanation of parameters see a).

$$C_n = \$5,000 \cdot \left(1 + \frac{318}{360} \cdot 0.039\right) \cdot (1 + 0.039)^6 \cdot$$
$$\cdot \left(1 + \frac{230}{360} \cdot 0.039\right)$$

$$C_n = \$6,669.01$$

Day of deposit → End of year	Period n-years	End of year → Day of payout
$\underbrace{\text{Remaining period 1}}_{\text{simple interest}}$	$\underbrace{\text{Interest period}}_{\text{compound computation of interest}}$	$\underbrace{\text{Remaining period 2}}_{\text{simple interest}}$
$\frac{30 \cdot 10}{360} + \frac{18}{360}$ $= \frac{318}{360} = n_{21}$ 10 months + 18 days Feb. 12th not included	6 years $= n_1$	$\frac{30 \cdot 7}{360} + \frac{20}{360}$ $= \frac{230}{360} = n_{22}$ 7 months + 20 days Aug. 20th included

With all ACT methods, the interest days are determined exactly to the calendar. Consequently, individual months are calculated with 30 or 31 interest days, or February with 28 or 29 interest days, depending on their actual number of days. Depending on the type of investment, interest is calculated either on the first or the last day of investment.

c) Actual/360 (Euro interest rate method)

Under the Actual/360 method, the interest days are divided by 360 to determine the proportion of the nominal annual interest rate. This results in 365 interest days for a full year or 366 interest days in a leap year. In the Euro interest rate method, interest is paid on the first day of investment; no interest is paid on the last day of investment.

$$C_n = C_0 \cdot \left(1 + \frac{\Delta t_1}{360} \cdot i\right) \cdot \left(1 + \frac{365}{360} \cdot i\right)^{t_2} \cdot$$

$$\cdot \left(1 + \frac{366}{360} \cdot i\right)^{t_3} \cdot \left(1 + \frac{\Delta t_4}{360} \cdot i\right)$$

C_n	Final Capital
C_0	Initial Capital
Δt_1	Period from capital contribution to 1st interest payment
Δt_4	Period from last interest payment to end of capital investment
t_2	Number of full years that are no leap years
t_3	Number of full leap years
i	Interest Rate p.a.
n	Period measured in years

$$C_n = \$5,000 \cdot \left(1 + \frac{323}{360} \cdot 0.039\right) \cdot \left(1 + \frac{365}{360} \cdot 0.039\right)^4 \cdot$$

$$\cdot \left(1 + \frac{366}{360} \cdot 0.039\right)^2 \cdot \left(1 + \frac{231}{360} \cdot 0.039\right)$$

$$C_n = \$6,695.50$$

Day of deposit → End of year	Period n-years	End of year → Day of payout
$\underbrace{\text{Remaining period 1}}_{\text{simple interest}}$	$\underbrace{\text{Interest period}}_{\text{compound computation of interest}}$	$\underbrace{\text{Remaining period 2}}_{\text{simple interest}}$
$\frac{17}{360}+\frac{4\cdot 30}{360}+\frac{6\cdot 31}{360}$ $=\frac{323}{360}=n_{21}$ 10 months + 17 days Feb. 12th included	4 years with 365 days per year $= n_{11}$ 2 years with 366 days per year $= n_{12}$	$\frac{28}{360}+\frac{2\cdot 30}{360}+\frac{4\cdot 31}{360}+$ $+\frac{19}{360}=\frac{231}{360}=n_{22}$ 7 months + 19 days Aug. 8th not included

d) Actual/360 (French interest rate method)

The only difference between the French interest rate method compared to the Euro interest rate method is that the first day of investment does not earn interest, but the last day of investment does.

$$C_n = C_0 \cdot \left(1+\frac{\Delta t_1}{360}\cdot i\right)\cdot\left(1+\frac{365}{360}\cdot i\right)^{t_2}\cdot$$

$$\cdot\left(1+\frac{366}{360}\cdot i\right)^{t_3}\cdot\left(1+\frac{\Delta t_4}{360}\cdot i\right)$$

For explanation of parameters see c).

$$
\begin{aligned}
C_n &= \$5,000 \cdot \left(1+\frac{322}{360}\cdot 0.039\right)\cdot\left(1+\frac{365}{360}\cdot 0.039\right)^4 \cdot \\
&\quad \cdot \left(1+\frac{366}{360}\cdot 0.039\right)^2 \cdot \left(1+\frac{232}{360}\cdot 0.039\right) \\
C_n &= \$6,695.51
\end{aligned}
$$

Day of deposit → End of year	Period n-years	End of year → Day of payout
$\underbrace{\text{Remaining period 1}}_{\text{simple interest}}$	$\underbrace{\text{Interest period}}_{\text{compound computation of interest}}$	$\underbrace{\text{Remaining period 2}}_{\text{simple interest}}$
$\frac{16}{360}+\frac{4\cdot 30}{360}+\frac{6\cdot 31}{360}$ $=\frac{322}{360}=n_{21}$	4 years with 365 days per year $= n_{11}$ 2 years with 366 days per year $= n_{12}$	$\frac{28}{360}+\frac{2\cdot 30}{360}+\frac{4\cdot 31}{360}+$ $+\frac{20}{360}=\frac{232}{360}=n_{22}$
10 months + 16 days		7 months + 20 days
Feb. 12th not included		Aug. 20th included

e) Actual/365 Fixed (English interest rate method)

This method involves dividing the interest days by 365 to determine the share of the nominal annual interest rate. This is the only difference from the Actual/360 method. No interest is calculated for the first day of investment while it is calculated for the last day of investment.

$$C_n = C_0 \cdot \left(1+\frac{\Delta t_1}{360}\cdot i\right)\cdot\left(1+\frac{365}{360}\cdot i\right)^{t_2}\cdot$$
$$\cdot\left(1+\frac{366}{360}\cdot i\right)^{t_3}\cdot\left(1+\frac{\Delta t_4}{360}\cdot i\right)$$

For explanation of parameters see c).

$$C_n = \$5,000\cdot\left(1+\frac{322}{365}\cdot 0.039\right)\cdot\left(1+\frac{365}{365}\cdot 0.039\right)^4\cdot$$
$$\cdot\left(1+\frac{366}{365}\cdot 0.039\right)^2\cdot\left(1+\frac{232}{365}\cdot 0.039\right)$$

$$C_n = \$6,669.26$$

Day of deposit → End of year	Period n-years	End of year → Day of payout
$\underbrace{\text{Remaining period 1}}_{\text{simple interest}}$	$\underbrace{\text{Interest period}}_{\text{compound computation of interest}}$	$\underbrace{\text{Remaining period 2}}_{\text{simple interest}}$
$\frac{16}{365}+\frac{4\cdot 30}{365}+\frac{6\cdot 31}{365}$ $=\frac{322}{365}=n_{21}$ 10 months + 16 days Feb. 12th not included	4 years with 365 days per year $= n_{11}$ 2 years with 366 days per year $= n_{12}$	$\frac{28}{365}+\frac{2\cdot 30}{365}+\frac{4\cdot 31}{365}+$ $+\frac{20}{365}=\frac{232}{365}=n_{22}$ 7 months + 20 days Aug. 20th included

f) Actual/Actual ICMA

The day-specific interest method provides that both the number of interest days and the length of the base year are always determined to the calendar. This results in 365 interest days for a full year or 366 interest days for a leap year.

$$C_n = C_0 \cdot \left(1 + \frac{\Delta t_1}{360} \cdot i\right) \cdot \left(1 + \frac{365}{360} \cdot i\right)^{t_2} \cdot$$

$$\cdot \left(1 + \frac{366}{360} \cdot i\right)^{t_3} \cdot \left(1 + \frac{\Delta t_4}{360} \cdot i\right)$$

For explanation of parameters see c).

$$C_n = \$5,000 \cdot \left(1 + \frac{322}{365} \cdot 0.039\right) \cdot \left(1 + \frac{365}{365} \cdot 0.039\right)^4 \cdot$$

$$\cdot \left(1 + \frac{366}{366} \cdot 0.039\right)^2 \cdot \left(1 + \frac{232}{365} \cdot 0.039\right)$$

$$C_n = \$6,667.89$$

Day of deposit → End of year	Period n-years	End of year → Day of payout
$\underbrace{\text{Remaining period 1}}_{\text{simple interest}}$	$\underbrace{\text{Interest period}}_{\text{compound computation of interest}}$	$\underbrace{\text{Remaining period 2}}_{\text{simple interest}}$
$\frac{16}{365}+\frac{4\cdot 30}{365}+\frac{6\cdot 31}{365}$ $=\frac{322}{365}=n_{21}$ 10 months + 16 days Feb. 12th not included	4 years with 365 days per year $= n_{11}$ 2 years with 366 days per year $= n_{12}$	$\frac{28}{365}+\frac{2\cdot 30}{365}+\frac{4\cdot 31}{365}+$ $+\frac{20}{365}=\frac{232}{365}=n_{22}$ 7 months + 20 days Aug. 20th included

7.1.3 Interest During the Period

(Sub-annual) parts of a year, usually a calendar year, are defined as interest period(s) (semi-annual, quarterly, monthly or daily interest rates).

m	is the number of sub-annual interest periods per year
j	is the relative periodic interest rate linearly distributed over the respective equally long interest periods during the year m

$$j=\frac{i}{m}$$

Interest during the year is calculated in the same way as the annual interest rate.

7.1.3.1 Simple Interest Calculation (linear)

Final Capital $C_n = C_0 \cdot (1 + n \cdot i) = C_0 \cdot (1 + N \cdot j)$

Initial Capital $C_0 = \dfrac{C_n}{1 + n \cdot i} = \dfrac{C_n}{1 + N \cdot j}$

Interest Rate $j = \dfrac{1}{N} \cdot \left(\dfrac{C_n}{C_0} - 1 \right)$

Period $N = \dfrac{1}{j} \cdot \left(\dfrac{C_n}{C_0} - 1 \right)$

Example: $C_0 = \$3,000$; interest rate in percentage$= 7\,(i = 0.07)$;

$N_1 = 5$ quarters; $N_2 = 0.3$ quarters

$j = \dfrac{7}{4} = 1.75\,\%$

$C_{5.3} = \$3,000 \cdot (1 + 5.3 \cdot 0.0175) = \$3,278.25$

with $j = 1.75\,\%$

7.1.3.2 Simple Interest Using the Nominal Annual Interest Rate

Nominal Interest Rate

$C_0 \cdot (1 + n \cdot i) = C_0 \cdot (1 + N \cdot j)$ with $N = m \cdot n$

$C_0 \cdot (1 + n \cdot i) = C_0 \cdot (1 + m \cdot n \cdot j)$

$i = m \cdot j$

Final Capital $C_n = C_0 \cdot (1 + n \cdot i)$

Initial Capital $C_0 = \dfrac{C_n}{1 + n \cdot i}$

Relative Interest Rate $j = \dfrac{1}{m \cdot n} \cdot \left(\dfrac{C_n}{C_0} - 1 \right)$

Period $n = \dfrac{1}{i} \cdot \left(\dfrac{C_n}{C_0} - 1 \right)$

Example: $C_0 = \$2{,}000;\ j = 1.25\,\%;\ n = 3.5;$

$m = 4,$ i.e. quarterly interest

Nominal Interest Rate $i = 4 \cdot 0.0125 = 0.05$

Final Capital $C_{3.5} = \$2{,}000 \cdot (1 + 3.5 \cdot 0.05) = \$2{,}350$

7.1.3.3 Compound Interest (exponential)

Final Capital $C_n = C_0 \cdot (1 + j)^n$

Initial Capital $C_0 = C_n \cdot (1 + j)^{-n}$

Interest Rate $j = \sqrt[n]{\dfrac{C_n}{C_0}} - 1$

Period $n = \dfrac{\ln\left(\dfrac{C_n}{C_0}\right)}{\ln(1 + j)}$

Example: $C_n = \$20,000;$

interest rate in percentage$= 7\,(i = 0.07);$

$m = 2$, i.e. semi-annual interest

Relative Periodic Interest Rate $j = \frac{7}{2} = 3.5\,\%$

Initial Capital $C_0 = \$20,000 \cdot (1 + 0.035)^{-15.5} = \$11,734.23$ with a relative periodic interest rate of $j = 3.5\,\%$

7.1.3.4 Interest with Compound Interest Using a Conforming Annual Interest Rate

A so-called conforming (periodic) interest rate $i_{conform}$ (hereinafter referred to as i_{con}) leads by definition to the same result as the annual interest rate i for m interest periods of less than one year.

Conforming Periodic Interest Rate

$$C_0 \cdot (1 + i_{con})^n = C_0 \cdot (1 + j)^N \qquad \text{with} \qquad N = m \cdot n$$

$$C_0 \cdot (1 + i_{con})^n = C_0 \cdot (1 + j)^{m \cdot n}$$

$$i_{con} = (1 + j)^m - 1$$

Final Capital $C_n = C_0 \cdot (1 + i_{con})^n$

Initial Capital $C_0 = C_n \cdot (1 + i_{con})^{-n}$

Interest Rate $j = \sqrt[m \cdot n]{\frac{C_n}{C_0}} - 1$

Period $$n = \frac{\ln\left(\frac{C_n}{C_0}\right)}{\ln(1+i_{con})}$$

Example: $C_0 = \$4,000;\ j = 0.5\,\%;\ m = 4;\ n = 6.5$

Conforming Periodic Interest Rate $$i_{con} = (1+0.005)^4 - 1 = 0.02015050063 \approx 2\,\%$$

Final Capital $$C_0 \cdot (1+i_{con})^n = C_0 \cdot (1+j)^{m \cdot n}$$

$$\$4,000 \cdot (1+0.02015050063)^{6.5} = \$4,553.84$$

$$\$4,000 \cdot (1+0.005)^{4 \cdot 6.5} = \$4,553.84$$

The present example demonstrates that a so-called conforming (periodic) interest rate i_{con} for m interest rates during the year leads by definition to the same result as the (sub-)annual interest rate j.

7.1.3.5 Mixed Interest

Final Capital $$C_n = C_0 \cdot (1+j)^{n_1} \cdot (1+n_2 \cdot j) \quad \text{with} \quad n_1 = int(n), \quad n_2 = n - n_1$$

$int(...)$ represents the integer function commonly used by pocket calculators. This means that n_1 is the largest number to which $n_1 \leq n$ applies. Consequently, n_2 is limited to the interval from 0 to 1, $n_2 \in [0,1]$.[1]

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[1] Cf. Kruschwitz, L. (2018): Finanzmathematik, 6th edition, p. 6.

Initial Capital $C_0 = \dfrac{C_n}{(1+j)^{n_1} \cdot (1+n_2 \cdot j)}$

Interest Rate Calculation of the zeros of the function

$$f(i) = -C_n + C_0 \cdot (1+j)^{n_1} \cdot (1+n_2 \cdot j)$$

Period $n = n_1 + \dfrac{1}{j} \cdot \left(\dfrac{C_n}{C_0 \cdot (1+j)^{n_1}} - 1\right)$

with $n_1 = int \dfrac{\left(\ln \dfrac{C_n}{C_0}\right)}{\ln(1+j)}$

Example: $C_0 = \$10,000$

interest rate in percentage $= 5\,\%$ p.a.

$n_1 = 12$ half-years; $n_2 = 3$ months $= 0.5$ half-years

$$j = \frac{0.05}{2} = 0.025$$

$$C_n = \$10,000 \cdot (1+0.025)^{12} \cdot (1+0.5 \cdot 0.025)$$

$$C_n = \$13,616.99$$

7.1.3.6 Steady Interest Rate

The steady interest rate is a special form of interest during the year, in which the number of interest periods m is infinite or converges towards infinity. The duration of an interest period is approaching zero.

Interest income is generated in infinitesimally short periods and accumulated to the (respective previous) capital. Capital and interest income are then (immediately) paid interest again (compound interest). Consequently, for a given nominal interest rate, the return of interest is higher with a steady interest rate than with a discrete interest rate (annual, semi-annual, etc.).

e	Euler's number $(2.71828...)$
n	Number (period) of interest in years
i	Interest rate p.a.

Final Capital

$$C_n = \lim_{m\to\infty}\left[C_0 \cdot \left(1+\frac{i}{m}\right)^{m\cdot n}\right]$$

$$= C_0 \cdot e^{i\cdot n}$$

Initial Capital $C_0 = C_n \cdot e^{-i\cdot n}$

Interest Rate $i = \dfrac{\ln(C_n)-\ln(C_0)}{n}$

Period $n = \dfrac{\ln(C_n)-\ln(C_0)}{i}$

Example: $C_0 = \$1,000$

interest rate in percentage $= 3.3\,\%$ p.a.

$n = 5.75$ years

Note: In this example, a so-called conforming (periodic) interest rate i_{con} was used. This by definition leads to the same result as the (sub-)annual interest rate j. Both interest rates were used as an example for semi-annual interest rates.

- with semi-annual interest rates

  using the
  (sub-)annual interest rate:

  $$j = \frac{0.033}{2} = 0.0165$$

  $$C_{5.75} = \$1,000 \cdot (1 + 0.0165)^{5.75 \cdot 2} \approx \$1,207.08$$

  using the
  conforming interest rate:

  $$i_{con} = \left(1 + \frac{0.033}{2}\right)^{2} - 1 \approx 0.033272$$

  $$C_{5.75}\$ = 1,000 \cdot (1 + 0.033272)^{5.75} \approx \$1,207.08$$

This example once again demonstrates that a so-called conforming (periodic) interest rate i_{con} with m sub-annual interest rates by definition leads to the same result as the (sub-)annual interest rate j.

- with quarterly interest rates

$$i_{con} = \left(1 + \frac{0.033}{4}\right)^{4} - 1 \approx 0.033411$$

$$C_{5.75} = \$1,000 \cdot (1 + 0.033411)^{5.75} \approx \$1,208.01$$

- with monthly interest rates

$$i_{con} = \left(1 + \frac{0.033}{12}\right)^{12} - 1 \approx 0.033504$$

$$C_{5.75} = \$1,000 \cdot (1 + 0.033504)^{5.75} \approx \$1,208.63$$

- with daily interest rates

  a) Actual/360

$$i_{con} = \left(1 + \frac{0.033}{360}\right)^{360} - 1 \approx 0.033549$$

$$C_{5.75} = \$1,000 \cdot (1 + 0.033549)^{5.75} \approx \$1,208.94$$

b) Actual/365, Actual/Actual

$$i_{con} = \left(1 + \frac{0.033}{365}\right)^{365} - 1 \approx 0.033549$$

$$C_{5.75} = \$1{,}000 \cdot (1 + 0.033549)^{5.75} \approx \$1{,}208.94$$

c) Actual/Actual (in case of a leap year)

$$i_{con} = \left(1 + \frac{0.033}{366}\right)^{366} - 1 = 0.033549$$

$$C_{5.75} = \$1{,}000 \cdot (1 + 0.033549)^{5.75} \approx \$1{,}208.94$$

- with steady interest rates

$$C_{5.75} = \$1{,}000 \cdot e^{0.033 \cdot 5.75} \approx \$1{,}208.95$$

The present example demonstrates that the daily interest rate comes very close to the steady interest rate in the result (capital end value after 5.75 years $C_{5.75}$), which is justified by the relatively short period of 5.75 years. On the other hand, the other periodic differences shown here are also significant, even for this short period.

7.2 Annual Percentage Rate

The annual percentage rate (APR) allows several credit offers with the same fixed interest periods to be compared. When calculating the effective annual interest rate, fees such as processing fees and discounts are included in addition to the nominal annual interest rate. Sometimes the APR corresponds to the nominal APR, the simple-interest rate for a year, and sometimes to the effective APR, the fee and compound interest rate calculated across a year.[2]

Effective Annual Percentage Rate

There is no exact legal definition of the effective APR. It depends on the type of fees inlcuded. The calculation of the APR can also be differentiated into at least three ways depending on if fees are added to the entire amount or treated as a short-term loan due in the first payment:

- Calculating the interest rate for each year without considering fees

  Example: loan: $\$200$; interest rate: $6\,\%$ p.a.; unique fee: $\$20$

  $$1.06^{12} = 2.0122 \approx 100\,\% \text{ increase}$$

- The origination fees are added on to the balance due; the total amount is treated as the basis for calculating compound interest

  Example: loan: $\$200$; interest rate: $6\,\%$ p.a.; unique fee: $\$20$

  $$\frac{\$20}{\$200} = 0.1 \qquad 0.1 + 0.06 = 0.16$$

  $$1.16^{12} = 5.9360 \approx 500\,\% \text{ increase}$$

- The origination fees are amortisation as a short-term loan. This loan becomes due with the first payment(s), and the unpaid balance

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[2] Cf. Wikimedia Foundation Inc. (Ed.) (2020): https://en.wikipedia.org/wiki/Annual_percentage_rate#cite_note-9, accessed 9 December 2022.

is amortised as a second long-term loan.The additional first payment(s) is intended mainly to pay the commitment fees and interest charges for that portion.[3]

United States

In the U.S., the calculation of APR is directed by the Truth in Lending Act, which is implemented by the Consumer Financial Protection Bureau (CFPB).[4] APR is expressed as a periodic interest rate times number of compounding periods during a year (e.g. semi-annual, quarterly, monthly, daily), which is also called the nominal interest rate.[5] The APR must include certain non-interest charges and fees. It has to be disclosed to the borrower within three days of applying for a mortgage. In the U.S. a distinction is made between a “close-ended credit” and an “open-ended credit”.

Close-ended Credit

In the U.S., a close-ended credit is a type of credit where the funds are distributed in full when the loan is terminated and the loan ends. It must be paid back by a specific date, including interest and finance charges. The loan may require the full payment of principal at maturity, or it may require regular principal and interest payments in defined periods. Close-ended credits are mainly used for home mortgages or auto loans.[6] For a fixed-rate mortgage, the APR is equal to the internal rate of return, if prepayment and default would be zero.

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[3] Cf. Wikimedia Foundation Inc. (Ed.) (2020): https://en.wikipedia.org/wiki/Annual_percentage_rate#cite_note-9, accessed 9 December 2022.

[4] Cf. Wikimedia Foundation Inc. (Ed.) (2020): https://en.wikipedia.org/wiki/Annual_percentage_rate#cite_note-9, accessed 9 December 2022.

[5] Cf. Tucker, W.R. (2000): Effective Interest Rate (EIR). In: Bankakademie Micro Banking Competence Center (Ed.): https://web.archive.org/web/20051103034219/http://www.uncdf.org/mfdl/readings/EIR_Tucker.pdf, accessed 13 October 2020.

[6] Cf. Federal Deposit Insurance Corporation (Ed.) (2014): https://www.fdic.gov/regulations/laws/rules/6500-3550.html, accessed 9 December 2022.

For an adjustable-rate mortgage the APR will also depend on the prospective trajectory of the index rate.[7]

Open-ended Credit

In the U.S., open-ended credit is a preapproved loan between a financial institution and a borrower that may be used up to a certain limit and can subsequently be paid back.[8] The preapproved amount will be defined in a formal agreement between the lender and the borrower.[9] Open-ended credits are mainly used for credit cards, home equity loans or other lines of credit.[10]

European Union

In the EU a single method of calculating the APR was introduced in 1998 (directive 98/7/EC), whose publication is needed for the major part of loans. Given the enhanced notation of directive 2008/48/EC, the basic equation for the calculation of APR in the EU is:

$$\sum_{i=1}^{M} C_i (1+\frac{APR}{100})^{-t_i} = \sum_{j=1}^{N} D_j (1+\frac{APR}{100})^{-s_j}$$

M	total number of drawdowns paid by the lender
N	total number of repayments paid by the borrower
i	sequence number of a drawdown paid by the lender
j	sequence number of a repayment paid by the borrower

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[7] Cf. Tucker, W.R. (2000): Effective Interest Rate (EIR). In: Bankakademie Micro Banking Competence Center (Ed.): https://web.archive.org/web/20051103034219/http://www.uncdf.org/mfdl/readings/EIR_Tucker.pdf, accessed 13 October 2020.

[8] Cf. Federal Deposit Insurance Corporation (Ed.) (2009): https://www.fdic.gov/regulations/laws/rules/6500-1650.html#6500226.14, accessed 9 December 2022.

[9] Cf. Twin, A. (2019): Open-End Credit. In: Investopedia (Ed.): https://www.investopedia.com/terms/o/openendcredit.asp, accessed 9 December 2022.

[10] Cf. Wikimedia Foundation Inc. (Ed.) (2020): https://en.wikipedia.org/wiki/Annual_percentage_rate#cite_note-9, accessed 9 December 2022.

C_i cash flow amount for drawdown number i

D_j cash flow amount for repayment number j

t_i interval, expressed in years and fractions of a year, between the date of the first drawdown and the date of drawdown i

s_j interval, expressed in years and fractions of a year, between the date of the first drawdown and the date of repayment j

- The EU formula makes use of the natural convention that all time intervals in t_i and s_j are measured relative to the date of the first drawdown, hence $t_1 = 0$. However, any other date could be used without affecting the calculated APR, as long as it is used consistently.
- The left side of this equation represents the present value of the drawdowns made by the lender and the right side shows the present value of the repayments made by the borrower.
- Neither the amounts nor the periods between transactions are necessarily equal. For the purpose of this computation it is assumed that a year has 365 days (366 days in a leap year), 52 weeks or 12 equal months.
- The result must be given with at least one decimal place.

Examples:

Example 1: different repayment amounts

Amount borrowed: $\$1,000$

Repayment: after 3 months: $\$274$

after 6 months: $\$274$

after 12 months: $\$548$

$$\$1,000 = \frac{\$274}{(1+\frac{APR}{100})^{\frac{3}{12}}} + \frac{\$274}{(1+\frac{APR}{100})^{\frac{6}{12}}} + \frac{\$548}{(1+\frac{APR}{100})^{\frac{12}{12}}}$$

$$1 + \frac{APR}{100} = q$$

$$\$1,000 = \frac{\$274}{q^{\frac{1}{4}}} + \frac{\$274}{q^{\frac{1}{2}}} + \frac{\$548}{q^{1}}$$

using numerical solution methods results in:

$$q \approx 1.1442283$$

$$i_{eff} = q - 1 \approx 1.1442283 - 1 \approx 0.1442283 \approx 14.42\,\%$$

Example 2: Discount and interest payments during the year

Amount borrowed: $\$5,000$ (finally due)
Discount: $10\,\%$
$n = 15$ months
$APR = 7.5\,\%$ (interest payments at the end and in the middle of each calendar year)

Interest payments: after 3 months (31.12.): $\$93.75$
after 9 months (01.07.): $\$187.50$
after 15 months (31.12.): $\$187.50$

$$\$4,500 = \frac{\$93.75}{(1+\frac{APR}{100})^{\frac{3}{12}}} + \frac{\$187.50}{(1+\frac{APR}{100})^{\frac{9}{12}}} + \frac{\$187.50 + \$5,000}{(1+\frac{APR}{100})^{\frac{15}{12}}}$$

$$1 + \frac{APR}{100} = q$$

$$\$4,500 = \frac{\$93.75}{q^{\frac{1}{4}}} + \frac{\$187.50}{q^{\frac{3}{4}}} + \frac{\$5,187.50}{q^{\frac{5}{4}}}$$

using numerical solution methods results in:

$$q \approx 1.1742722$$

$$i_{eff} = q - 1 \approx 1.1742722 - 1 \approx 0.1742722 \approx 17.43\,\%$$

7.3 Depreciation

Impairments are recognized in the accounts through depreciation over their economic life. This concerns fixed and current assets. A distinction is made between time depreciation and performance depreciation.

7.3.1 Time Depreciation

The acquisition costs or production costs are distributed among the years of the economic life.

7.3.1.1 Linear Depreciation

A	original cost or par
n	economic life in years
Q_k	amount of depreciation, by which the book value is reduced in the k^{th} year
R_k	book value after k years (with $k = 1, 2, 3, ..., n$) $R_k = A - \sum Q_k$
R_n	residual value (salvage value, old value, scrap value) at the end of the economic life
i	depreciation rate

With linear depreciation, the difference between acquisition or production costs and the residual value at the end of the economic life is distributed evenly over the periods of use. It is assumed that the value is consumed evenly over the useful life.

The following applies: $Q_1 = Q_2 = ... = Q_n = \frac{A - R_n}{n}$

Example: A company acquires a vehicle with a value of \$90,000, assuming an economic life of 9 years. It is also assumed that the vehicle can be sold for \$9,000 at the end of its useful life. The company opts for linear depreciation.

$$A = \$90{,}000;\ \ R_n = \$9{,}000;\ \ n = 9$$

amount of depreciation:

$$Q_1 = \frac{\$90{,}000 - \$9{,}000}{9} = \$9{,}000$$

depreciation rate:

$$i = \frac{\$9{,}000}{\$90{,}000 - \$9{,}000} \cdot 100\% = 11.11\,\%$$

7.3.1.2 Arithmetic-Degressive Depreciation

With arithmetic-degressive depreciation, the annual depreciation amounts are reduced by a constant amount d. Thus, the first years are more heavily burdened than the later ones. This results in an assumption of decreasing depreciation over the economic life.

d	degressive amount
N	total of the years' numbers of the economic life
T_k	remaining useful life at the beginning of the year after k years

Degressive Amount

$$d = \frac{\text{original cost} - \text{residual value}}{\text{total of the years' numbers}}$$

or $d = \dfrac{A - R_n}{1 + 2 + 3 + ... + n}$

or $d = \dfrac{A - R_n}{\dfrac{n \cdot (n+1)}{2}}$

or $d = \dfrac{2 \cdot (A - R_n)}{n \cdot (n+1)}$

or $d = \dfrac{A - R_n}{N}$

Amount of Depreciation $Q_k = d \cdot T_k$

Example: A company acquires a vehicle with a value of $90,000, assuming an economic life of 9 years. It is also assumed that the vehicle can be sold for $9,000 at the end of its useful life. It is assumed that the depreciation will decrease by a constant amount.

$$A = \$90,000; \;\; R_n = \$9,000; \;\; n = 9$$

$$d = \frac{\$90,000 - \$9,000}{1 + 2 + 3 + 4 + ... + 9} = \$1,800$$

$$Q_1 = \$1,800 \cdot 9 = \$16,200$$

$$Q_2 = \$1,800 \cdot 8 = \$14,400$$

$$Q_3 = \$1,800 \cdot 7 = \$12,600$$

$$\vdots$$

$$Q_9 = \$1,800 \cdot 1 = \$1,800$$

The example illustrates that the degressive amount of $1,800 corresponds to the amount of depreciation in the last year of the economic life.

7.3.1.3 Geometric-Degressive Depreciation

With geometric-degressive depreciation, the annual amounts of depreciation are reduced by the depreciation rate.

Determination of the book values R_k and the residual value R_n

Beginning of the 1st year A

End of the 1st year $R_1 = A - A \cdot i = A \cdot (1-i)$

End of the 2nd year $R_2 = R_1 - R_1 \cdot i = R_1 \cdot (1-i) = A \cdot (1-i)^2$

End of the 3rd year $R_3 = R_2 - R_2 \cdot i = R_2 \cdot (1-i) = A \cdot (1-i)^3$

⋮

End of the nth year $R_n = R_{n-1} - R_{n-1} \cdot i = R_{n-1} \cdot (1-i)$

$$R_n = A \cdot (1-i)^n$$

Determination of the amounts of depreciation Q_k

End of the 1st year $Q_1 = A \cdot i$

End of the 2nd year $Q_2 = R_1 \cdot i = A \cdot (1-i) \cdot i$

End of the 3rd year $Q_3 = R_2 \cdot i = A \cdot (1-i)^2 \cdot i$

⋮

End of the nth year $Q_n = R_{n-1} \cdot i = A \cdot (1-i)^{n-1} \cdot i$

Determination of the depreciation rate $i = \frac{p}{100}$

The depreciation rate i is determined by the ratio of the target residual value R_n to the original value A.

$$\begin{aligned}
& R_n = A \cdot \left(1 - \frac{p}{100}\right)^n && \mid \div A \\
\Leftrightarrow \quad & \frac{R_n}{A} = \left(1 - \frac{p}{100}\right)^n && \mid \sqrt[n]{\ldots} \\
\Leftrightarrow \quad & \sqrt[n]{\frac{R_n}{A}} = 1 - \frac{p}{100} && \mid -1; \ \cdot(-100) \\
\Leftrightarrow \quad & p = 100 \cdot \left(1 - \sqrt[n]{\frac{R_n}{A}}\right)
\end{aligned}$$

Example: A company acquires a vehicle with a value of \$90,000, assuming an economic life of 9 years. It is also assumed that the vehicle can be sold for \$9,000 at the end of its useful life. The amount of depreciation shall decrease annually by a constant depreciation rate.

$$A = \$90{,}000; \ \ R_n = \$9{,}000; \ \ n = 9$$

$$p = 100 \cdot \left(1 - \sqrt[9]{\frac{\$9{,}000}{\$90{,}000}}\right) \approx 22.57\,\%$$

$$Q_1 = \$90{,}000.00 \cdot 0.2257 = \$20{,}313.00$$

$$R_1 = \$90{,}000.00 - 20{,}313.00 = \$69{,}687.00$$

$$Q_2 = \$69{,}687.00 \cdot 0.2257 = \$15{,}728.36$$

$$R_2 = \$69{,}687.00 - \$15{,}728.36 = \$53{,}958.64$$

k	Q_k in $	R_k in $
1	20,313.00	69,687.00
2	15,728.36	53,958.64
3	12,178.47	41,780.17
4	9,427.78	32,352.39
5	7,301.93	25,050.46
6	5,653.89	19,396.57
7	4,377.81	15,018.76
8	3,389.73	11,629.03
9	2,624.67	9,004.36

Note: Differences are due to rounding errors.

7.3.2 Units of Production Depreciation

In accordance with the changing use of the assets, depreciation is made according to the intensity of use. The amount of depreciation for a period depends on the performance consumed in that period. Therefore, there can usually be no uniform trend for the course of the annual amounts of depreciation or a constant rate of depreciation.

P_A	total performance of the asset
P_{Pt}	performance consumed during the period

Example: A company acquires a vehicle worth $90,000, assuming a total performance of 300,000 miles. In the first year, the vehicle covers 50,000 miles. It should be depreciated according to consumption.

$$A = \$90,000; \;\; P_A = 300,000 \text{ miles}; \;\; P_{Pt} = 50,000 \text{ miles}$$

$$Q_k = \frac{50,000}{300,000} \cdot \$90,000 = \$15,000$$

7.3.3 Extraordinary Depreciation

In addition to the previously explained scheduled depreciation methods which record a permanent decline in value, extraordinary depreciation can also be applied. Unscheduled or extraordinary depreciation records impairment losses that are not caused by the planned, assumed use. This is the case, for example, in the event of extraordinary technological progress or in the event of unforeseeable, i.e. unplanned damage to property.

Example: A company acquires a vehicle valued at $90,000. After the first two years of depreciation, the vehicle has a book value of $72,000. In the third year, the vehicle is involved in an accident and is severely damaged. Despite repairs, the vehicle has lost value as a car that was involved in an accident. An appraiser certifies a current value of $40,000.

The regular depreciation is based on an economic life of 9 years and a selling price of $9,000 at the end of the useful life.

Regular amount of depreciation:

$$Q_1 = \frac{\$90,000 - \$9,000}{9} = \$9,000$$

Extraordinary amount of depreciation:

$$\$72,000 - \$40,000 = \$32,000$$

Instead of the scheduled depreciation amount of \$9,000, a depreciation amount of \$32,000 is applied in the third year of the useful life. This means that the vehicle is in the books of the company with a current (book) value of \$40,000.

7.4 Annuity Calculation

7.4.1 Fundamental Terms

An annuity r is a recurring payment made or received at regular intervals. The payments can be either deposits or disbursements.

Present Value of Annuity R_0	Total value of an annuity at the beginning of the payment period
Amount of Annuity R_n	Total value of an annuity after n years
Annuity r	Regularly paid instalment
Interest Factor q	Annual interest factor $q = 1 + i$
Accumulation Factor	Accumulates interest on a monetary amount exponentially (with interest and compound interest) over n periods. $q^n = (1+i)^n$
Discount Factor	Discounts a monetary amount exponentially (with interest and compound interest) over n periods.

$$q^{-n} = (1+i)^{-n}$$

Annuity Present Value Factor

$$\frac{q^n - 1}{q^n \cdot (q-1)} = \frac{(1+i)^n - 1}{(1+i)^n \cdot i}$$

With the aid of the annuity present value factor, the present value of annuities of uniform (annuity) payments can be determined.

(a) (annually) in arrears:

$$R_0 = r \cdot \frac{q^n - 1}{q^n \cdot (q-1)} = r \cdot \frac{q^n - 1}{q^n \cdot i}$$

(b) (annually) in advance:

$$R_0 = r \cdot \frac{q \cdot (q^n - 1)}{q^n \cdot (q-1)} = r \cdot \frac{q \cdot (q^n - 1)}{q^n \cdot i}$$

Final Annuity Value Factor

$$\frac{q^n - 1}{q - 1} = \frac{(1+i)^n - 1}{i}$$

With the aid of the final annuity value factor, the amount of annuity of uniform (annuity) payments can be determined.

(a) (annually) in arrears:

$$R_n = r \cdot \frac{q^n - 1}{q - 1} = r \cdot \frac{q^n - 1}{i}$$

(b) (annually) in advance:

$$R_n = r \cdot \frac{q \cdot (q^n - 1)}{q - 1} = r \cdot \frac{q \cdot (q^n - 1)}{i}$$

Annuity Factor

The annuity factor distributes a fixed amount of money at equal annuities A, taking into account interest and compound interest, over n periods. The annuity factor therefore corresponds to the reciprocal of the present value factor.

(a) (annually) in arrears

$$A_{arrears} = \frac{q^n \cdot (q-1)}{q^n - 1} = \frac{q^n \cdot i}{q^n - 1}$$

(b) (annually) in advance

$$A_{advance} = \frac{q^n \cdot i}{q \cdot (q^n - 1)}$$

Example:

Mrs. Penny will inherit $1,000,000 on January 1st, 2010. She would like to consume this amount in equal parts every year for the next 15 years. In doing so, she calculates with a calculation interest rate (average interest rate during these 15 years) of $2.5\,\%$ p.a. Mrs. Penny would like to have the yearly annuity paid out at the beginning of each (calendar) year.

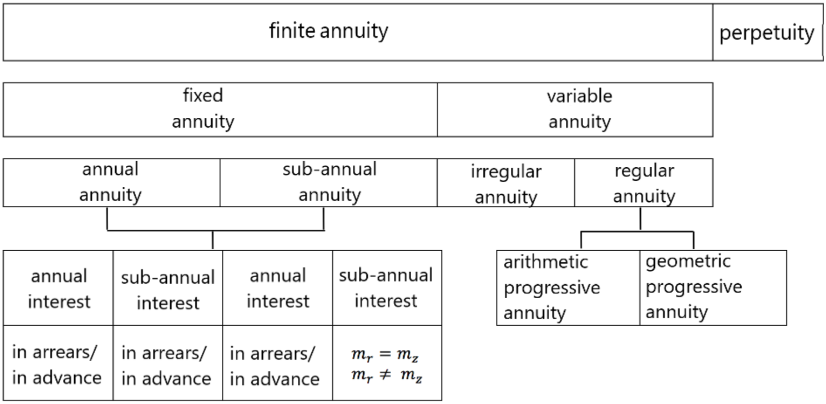
This annual, in advance annuity factor is calculated as follows

$$A_{advance} = \frac{q^n \cdot i}{q \cdot (q^n - 1)} = \frac{(1.025)^{15} \cdot 0.025}{(1.025) \cdot (1.025^{15} - 1)} \approx 0.078797$$

$$\Rightarrow R_{15}^{adv.} \approx \$78,797$$

Mrs. Penny has around $78,797 available at the beginning of each (calendar) year for 15 years, if she would like to consume the sum of $1,000,000 as planned at a calculation interest rate of $2.5\,\%$ p.a.

Overview:



7.4.2 Finite, Regular Annuity

7.4.2.1 Annual Annuity with Annual Interest

Annuity and interest periods are exactly one year.

(a) Annuity in arrears: Payment due at the end of the year

Amount of Annuity

$$R_1 = r_1$$

$$R_2 = r_2 + R_1 \cdot q = r_2 + r_1 \cdot q$$

$$R_3 = r_3 + R_2 \cdot q = r_3 + r_2 \cdot q + r_1 \cdot q^2$$

$$\vdots$$

$$R_n = r_n + r_{n-1} \cdot q + r_{n-2} \cdot q^2 + ... + r_2 \cdot q^{n-2} + r_1 \cdot q^{n-1}$$

$$R_n = r \cdot (1 + q^1 + q^2 + ... + q^{n-1})$$

$$R_n = r \cdot \underbrace{\frac{q^n - 1}{q - 1}}_{\text{final annuity value factor}} = r \cdot \frac{q^n - 1}{i}$$

Present Value of Annuity

$$C_n = C_0 \cdot q^n \ \widehat{=} \ R_n = R_0 \cdot q^n$$

$$R_0 = r \cdot \underbrace{\frac{q^n - 1}{q^n \cdot (q - 1)}}_{\text{annuity present value factor}} = r \cdot \frac{q^n - 1}{q^n \cdot i}$$

Annuity (R_n given)

$$r = R_n \cdot \frac{q - 1}{q^n \cdot (q - 1)} = R_n \cdot \frac{i}{q^n - 1}$$

Annuity (R_0 given)

$$r = R_0 \cdot \frac{(q - 1) \cdot q^n}{q^n - 1} = R_0 \cdot \frac{i \cdot q^n}{q^n - 1}$$

Interest Rate (R_n given)

$$R_n = r \cdot \frac{q^n - 1}{i}$$

Calculation of the zeros of the function

$$f(i) = -R_n + r \cdot \frac{(1 + i)^n - 1}{i}$$

$$f'(i) = r \cdot \frac{i \cdot n \cdot q^{n-1} - q^n + 1}{i^2}$$

Newton's method of approximation

$$i_{k+1} = i_k - \frac{f(i_k)}{f'(i_k)}$$

Interest Rate (R_0 given)

$$R_0 = r \cdot \frac{q^n - 1}{q^n \cdot (q-1)}$$

Calculation of the zeros of the function

$$f(i) = -R_0 + r \cdot \frac{(1+i)^n - 1}{i \cdot (1+i)^n}$$

$$f'(i) = r \cdot \frac{q + n \cdot i - q^{n+1}}{i^2 \cdot q^{n+1}}$$

Newton's tangent method

$$i_{k+1} = i_k - \frac{f(i_k)}{f'(i_k)}$$

Period (R_n given)

$$R_n = r \cdot \frac{q^n - 1}{i} \qquad | \cdot i;\ \div r;\ +1$$

$$q^n = 1 + \frac{i \cdot R_n}{r} \qquad | \ln(\ldots)$$

$$n = \frac{\ln\left(1 + \frac{i \cdot R_n}{r}\right)}{\ln(q)}$$

Period (R_0 given)

$$R_0 = r \cdot \frac{q^n - 1}{i \cdot q^n}$$

$$n = \ln\left(\frac{i \cdot R_n}{r} + 1\right) \cdot \frac{1}{\ln(q)}$$

(b) Annuity in advance: Payment due at the beginning of the year

Amount of Annuity

$$R_1 = r_0 \cdot q$$

$$R_2 = r_1 \cdot q + R_1 \cdot q = r_1 \cdot q^1 + r_0 \cdot q^2$$

$$R_3 = r_2 \cdot q + R_2 \cdot q = r_2 \cdot q^1 + r_1 \cdot q^2 + r_0 \cdot q^3$$

$$\vdots$$

$$R_n = r_{n-1} \cdot q^1 + rn - 2 \cdot q^2 + ... + r_2 \cdot q^{n-2} + r_1 \cdot q^{n-1}$$

$$+ r_0 \cdot q^n$$

$$R_n = r \cdot (q^1 + q^2 + ... + q^n)$$

$$R_n = r \cdot q \cdot (1 + q^1 + q^2 + ... + q^{n-1})$$

$$R_n = r \cdot q \cdot \frac{q^n - 1}{q - 1} = r \cdot q \cdot \frac{q^n - 1}{i}$$

Present Value of Annuity

$$R_0 = r \cdot \frac{q \cdot (q^n - 1)}{i \cdot q^n}$$

Annuity (R_n given)

$$r = R_n \cdot \frac{q - 1}{q \cdot (q^n - 1)} = \frac{R_n \cdot i}{q \cdot (q^n - 1)}$$

Annuity (R_0 given)

$$r = R_0 \cdot \frac{(q - 1) \cdot q^n}{q \cdot (q^n - 1)} = R_0 \cdot \frac{i \cdot q^n}{q \cdot (q^n - 1)}$$

Interest Rate (R_n given)

$$f(i) = -R_n + r \cdot \frac{(1 + i)^{n+1} - (1 + i)}{i}$$

$$f'(i) = r \cdot \frac{i \cdot (n + 1) \cdot q^n - q^{n+1} + 1}{i^2}$$

Interest Rate (R_0 given)

$$f(i) = -R_0 + r \cdot \frac{(1 + i)^{n+1} - (1 + i)}{i \cdot (1 + i)^n}$$

$$f'(i) = r \cdot \frac{i \cdot ((n+1) \cdot n) - q + q^{1-n}}{i^2}$$

Period (R_n given) $$n = \frac{\ln\left(q + \frac{i \cdot R_n}{r}\right)}{\ln(q)} - 1$$

Period (R_0 given) $$n = 1 - \frac{\ln\left(q - \frac{i \cdot R_0}{r}\right)}{\ln(q)}$$

7.4.2.2 Annual Annuity with Sub-Annual Interest

The annuity periods cover one year ($m_r = 1$), however there are multiple interest periods per year ($m_z > 1$).

Nominal, relative and conforming interest rate

Conforming Interest Rate discrete $$i^* = (1+j)^{m_z} - 1 = \left(1 + \frac{i}{m_z}\right)^{m_z} - 1$$

steady $$i^* = e^i - 1$$

Relative Interest Rate $$j = (1+i^*)^{\frac{1}{m_z}} - 1$$

(a) Annuity in arrears:

Amount of Annuity $$R_n = r \cdot \frac{(1+i^*)^n - 1}{i^*}$$

Present Value of Annuity $$R_0 = r \cdot \frac{(1+i^*)^n - 1}{i^* \cdot (1+i^*)^n}$$

Annuity (R_n given) $$r = R_n \cdot \frac{i^*}{(1+i^*)^n - 1}$$

Annuity (R_0 given) $$r = R_n \cdot \frac{i^* \cdot (1+i^*)^n}{(1+i^*)^n - 1}$$

Interest Rate (R_n given) $$f(i^*) = -R_n + r \cdot \frac{(1+i^*)^n - 1}{i^*}$$

$$f'(i^*) = r \cdot \frac{i^* \cdot n \cdot (1+i^*)^{n-1} - (1+i^*)^n + 1}{(i^*)^2}$$

Interest Rate (R_0 given) $$f(i^*) = -R_0 + r \cdot \frac{(1+i^*)^n - 1}{i^* \cdot (1+i^*)^n}$$

$$f'(i^*) = r \cdot \frac{(1+i^*) + n \cdot i^* - (1+i^*)^{n+1}}{(i^*)^2 \cdot (1+i^*)^{n+1}}$$

Period (R_n given) $$n = \frac{\ln\left(1 + \dfrac{i^* \cdot R_n}{r}\right)}{\ln(1+i^*)}$$

Period (R_0 given) $$n = \ln\left(\frac{i^* \cdot R_n}{r} + 1\right) \cdot \frac{1}{\ln(q)}$$

<u>Example:</u>

An annuity of \$700 payable annually in arrears is paid over a period of eight years. The interest rate is $1.25\,\%$ per quarter. What is the present value?

$$i^* = (1+0.0125)^4 - 1 = \left(1 + \frac{0.05}{4}\right) - 1 \approx 0.05095$$

$$R_0 = \$700 \cdot \frac{(1+0.05095)^8 - 1}{0.05095 \cdot (1+0.05095)^8} = \$4,506.93$$

The present value is \$4,506.93.

(b) Annuity in advance:

Amount of Annuity

$$R_n = r \cdot \frac{(1+i^*)^{n+1} - (1+i^*)}{i^*}$$

Present Value of Annuity

$$R_0 = r \cdot \frac{(1+i^*)^n - 1}{i^* \cdot (1+i^*)^{n-1}}$$

Annuity (R_n given)

$$r = R_n \cdot \frac{i^*}{(1+i^*)^{n+1} - (1+i^*)}$$

Annuity (R_0 given)

$$r = R_0 \cdot \frac{i^* \cdot (1+i^*)^{n-1}}{(1+i^*)^n - 1}$$

Interest Rate (R_n given)

$$f(i^*) = -R_n + r \cdot \frac{(1+i^*)^{n+1} - (1+i^*)}{i^*}$$

$$f'(i^*) = r \cdot \frac{(n+1)\cdot(1+i^*)^n \cdot i^* - i^* - (1+i^*)^{n+1} + (1+i^*)}{(i^*)^2}$$

Interest Rate (R_0 given)

$$f(i^*) = -R_0 + r \cdot \frac{(1+i^*)^n - 1}{i^* \cdot (1+i^*)^{n-1}}$$

$$f'(i^*) = r \cdot \frac{n \cdot i^* - (1+i^*) - i^* \cdot (n-1)}{(i^*)^2}$$

Period (R_n given)

$$n = \frac{\ln\left((1+i^*) + \dfrac{i^* \cdot R_n}{r}\right)}{\ln(1+i^*)} - 1$$

Period (R_0 given)

$$n = 1 - \frac{\ln\left((1+i^*) - \dfrac{i^* \cdot R_0}{r}\right)}{\ln(^{+}i^*)}$$

Example:

An annual annuity of \$700 payable in advance is paid over a period of eight years. The interest rate is $1.25\,\%$ per half year. What is the amount of annuity?

$$i^* = (1+0.0125)^2 - 1 = \left(1+\frac{0.025}{2}\right)^2 - 1 \approx 0.02516$$

$$R_8 = \$700 \cdot \frac{(1+0.02516)^{8+1} - (1+0.02516)}{0.02516} = \$4,585.76$$

The amount of annuity after eight years is \$4,585.76.

7.4.2.3 Sub-Annual Annuity with Annual Interest

Annuities are paid in sub-annual annuity periods (semi-annual $m_r = 2$, quarterly $m_r = 4$, monthly $m_r = 12$), the interest periods cover one year.

T is equivalent to the amount of the regular annuity payments

(a) Integer periods

Annuity payment in arrears

$$T = r \cdot \left(m_r + \frac{i}{m_r} \cdot \Big[0+1+2+\ldots+(m_r-1)\Big]\right)$$

with $0+1+2+\ldots+(m_r-1) = \dfrac{(m_r-1)\cdot m_r}{2}$

$$T = r \cdot \left(m_r + \frac{i}{m_r} \cdot \left[\frac{(m_r-1)\cdot m_r}{2}\right]\right)$$

$$T = r \cdot \left(m_r + \frac{i}{2} \cdot (m_r - 1) \right)$$

Amount of Annuity $$R_n = \underbrace{r \cdot \left(m_r + \frac{i}{2} \cdot (m_r - 1) \right)}_{T} \cdot \frac{q^n - 1}{i}$$

Present Value of Annuity $$R_0 = r \cdot \left(m_r + \frac{i}{2} \cdot (m_r - 1) \right) \cdot \frac{q^n - 1}{i \cdot q^n}$$

Annuity payment in advance

$$T = r \cdot \left(m_r + \frac{i}{m_r} \cdot \left[1 + 2 + 3 + \ldots + m_r \right] \right)$$

with $1 + 2 + 3 + \ldots + m_r = \dfrac{(m_r + 1) \cdot m_r}{2}$

$$T = r \cdot \left(m_r + \frac{i}{m_r} \cdot \left[\frac{(m_r + 1) \cdot m_r}{2} \right] \right)$$

$$T = r \cdot \left(m_r + \frac{i}{2} \cdot (m_r + 1) \right)$$

Amount of Annuity $$R_n = \underbrace{r \cdot \left(m_r + \frac{i}{2} \cdot (m_r + 1) \right)}_{T} \cdot \frac{q^n - 1}{i}$$

Present Value of Annuity $$R_0 = r \cdot \left(m_r + \frac{i}{2} \cdot (m_r + 1) \right) \cdot \frac{q^n - 1}{i \cdot q^n}$$

Example:

Mrs. Penny inherits a monetary amount, which she invests at an interest rate of $4\,\%$ p.a. She would like to be paid a constant monthly amount of \$1,400 in advance for ten years, so that the money is used up at the end of the period. How much money did she inherit?

$$R_0 = \$1,400 \cdot \left(12 + \frac{0.04}{2} \cdot (12+1)\right) \cdot \frac{1.04^{10} - 1}{0.04 \cdot 1.04^{10}} = \$139,215.42$$

The amount of money that Mrs. Penny inherited is \$139,215.42.

(b) Non-integer periods

$$n = \frac{N}{m_r} \qquad N \in \mathbb{Z}$$

$$N_1 = n_1 \cdot m_r \qquad n_1 = int(n)$$

and with

$$N_2 = n_2 \cdot m_r \qquad n_2 = n - n_1$$

Annuity payment in arrears

Amount of Annuity

$$R_n = r \cdot \left[\left(m_r + \tfrac{i}{2} \cdot (m_r - 1)\right) \cdot \tfrac{q^{n_1}-1}{i} \cdot (1 + n_2 \cdot i) + \left(N_2 + \tfrac{1}{m_r} \cdot \tfrac{(N_2-1)\cdot N_2}{2}\right)\right]$$

Present Value of Annuity

$$R_0 = r \cdot \frac{\left(m_r + \frac{i}{2} \cdot (m_r - 1)\right) \cdot \frac{q^{n_1} - 1}{i} \cdot (1 + n_2 \cdot i) + \left(\frac{1}{m_r} \cdot \frac{(N_2 - 1) \cdot N_2}{2}\right)}{q^{n_1} \cdot (1 + n_2 \cdot i)}$$

Annuity payment in advance

Amount of Annuity

$$R_n = r \cdot \left[\left(m_r + \tfrac{i}{2} \cdot (m_r + 1) \right) \cdot \tfrac{q^{n_1} - 1}{i} \cdot (1 + n_2 \cdot i) + \left(N_2 + \tfrac{1}{m_r} \cdot \tfrac{(N_2 + 1) \cdot N_2}{2} \right) \right]$$

Present Value of Annuity

$$R_0 = r \cdot \frac{\left(m_r + \frac{i}{2} \cdot (m_r + 1) \right) \cdot \frac{q^{n_1} - 1}{i} \cdot (1 + n_2 \cdot i) + \left(\frac{1}{m_2} \cdot \frac{(N_2 + 1) \cdot N_2}{2} \right)}{q^{n_1} \cdot (1 + n_2 \cdot i)}$$

Example:

Mrs. Penny deposits an amount of $\$500$ each quarter in arrears into an account. The interest rate is $2\,\%$ p.a. How much money is in the account after 10 years and 6 months?

$$N_2 = 0.5 \cdot 4 = 2$$

$$\begin{aligned} R_{10.5} = \$500 \cdot & \left[\left(12 + \frac{0.02}{2} \cdot (12 - 1) \right) \cdot \frac{1.02^{10} - 1}{0.02} \cdot (1 + 0.5 \cdot 0.02) \right. \\ & \left. + \left(2 + \frac{1}{12} \cdot \frac{(2 - 1) \cdot 2}{2} \right) \right] \end{aligned}$$

$$R_{10.5} = \$68,005.23$$

After 10 years and 6 months there are \$68,005.23 in the account.

7.4.2.4 Sub-Annual Annuity with Sub-Annual Interest

Annuity and interest periods are shorter than one year.

(a) Annuity period = interest period

Annuity payment in arrears

Amount of Annuity $$R_N = r \cdot \frac{q^N - 1}{j} = r \cdot \frac{(1+j)^N - 1}{j} = r \cdot \frac{q^N - 1}{j}$$

Present Value of Annuity $$R_0 = r \cdot \frac{q^N - 1}{q^N \cdot (q-1)} = r \cdot \frac{(1+j)^N - 1}{(1+j)^N \cdot j} = r \cdot \frac{q^N - 1}{j \cdot q^N}$$

Annuity payment in advance

Amount of Annuity $$R_N = r \cdot q \cdot \frac{q^N - 1}{j}$$

Present Value of Annuity $$R_0 = r \cdot \frac{q \cdot (q^N - 1)}{j \cdot q^N}$$

with $j = \frac{i}{m_z}$ sub-annual interest rate

$N = m_r \cdot n$ period of annuity in the sub-annual periods

$n =$ number of relevant years

$m_z =$ number of sub-annual interest periods

$m_r =$ number of sub-annual annuity periods

here $m_r = m_z$, since annuity periods $=$ interest rates

Example 1:

Camilla saves an amount of \$100 at the end of each month, which earns interest at 1.2% p.a. per month. What is her capital after 1.5 years?

$R_N = $?

with $r = \$100 \quad j = \frac{0.012}{12} = 0.001 \quad N = 12 \cdot 1.5 = 18 \quad q = 1.001$

$$\Rightarrow R_N = 100 \cdot \frac{(1.001)^{18} - 1}{0.001} = \$1,815.38$$

Example 2:

Michelle inherits \$20,000, which earns interest at her commercial bank at 3% p.a. per month. She wants to withdraw a constant amount at the beginning of each month for two years. How much is this monthly annuity payment in advance?

$r = $?

$$R_0 = r \cdot \frac{q \cdot (q^N - 1)}{j \cdot q^N}$$

$$\Rightarrow r = R_0 \cdot \frac{j \cdot q^N}{q \cdot (q^N - 1)}$$

with $R_0 = \$20,000$

$$j = \frac{0.03}{12} = 0.0025$$

$$q = 1.0025$$

$$N = 12 \cdot 2 = 24$$

$$r = 20,000 \cdot \frac{0.0025 \cdot 1.0025^{24}}{1.0025 \cdot (1.0025^{24} - 1)} = \$857.48$$

Example 3:

Steven wants to deposit an amount of \$500 into his account at the beginning of each month, which will earn interest at 0.25% each month. His goal is to have \$10,000 at the end of the period of this annuity. How long does Steven have to pay in?

$n =$?

$$R_N = r \cdot \frac{q \cdot (q^N - 1)}{j} \quad \text{(in advance)}$$

with $N = m_r \cdot n$

$$\Rightarrow R_N = r \cdot \frac{q \cdot (q^{m_r \cdot n} - 1)}{j}$$

$$\Leftrightarrow \frac{R_N}{r} \cdot j = q \cdot (q^{m_r \cdot n} - 1)$$

$$\Leftrightarrow \frac{R_N \cdot j}{r \cdot q} + 1 = q^{m_r \cdot n}$$

$$\Leftrightarrow ln\left(\frac{R_N \cdot j}{r \cdot q} + 1\right) = m_r \cdot n \cdot lnq$$

$$\Leftrightarrow n = \frac{ln\left(\frac{R_N \cdot j}{r \cdot q} + 1\right)}{lnq \cdot m_r}$$

with $R_N = \$10,000$

$r = \$500$

$m_z = 0.0025$ bzw. $j = 0.0025 \cdot 12 = 0.03 = 3\,\%$ p.a.

$q = 1.0025$ $N = m_r \cdot n$ with $m_r = 12$

$N =$ period in months

$n =$ period in years

$$n = \frac{ln\left(\frac{10,000 \cdot 0.0025}{500 \cdot 1.0025} + 1\right)}{ln 1.0025 \cdot 12} \approx 1.6244 \text{ Jahre}$$

$N = 12 \cdot 1.6244 \approx 19.4929$ months

(b) Annuity period < interest period

Annuity payment in arrears

Amount of Annuity $$R_N = r \cdot \frac{(1+j^*)^N - 1}{j^*} = r \cdot \frac{\left(1 + \frac{i^*}{m_r}\right)^N - 1}{\frac{i^*}{m_r}}$$

with $i^* =$ conforming interest rate

$$j^* = \frac{i^*}{m_r}$$

discrete $i^* = (1+j)^{m_z} - 1 = \left(1 + \frac{i}{m_z}\right)^{m_z} - 1$

continuous $i^* = e^i - 1$

and $j =$ relative interest rate

$$j = \frac{1}{m_z}$$

or $j^* = (1+j)^{m_z/m_r} - 1$

and $N = m_r \cdot n$

Example:

Nawid deposits an amount of \$300 into a savings account at the end of each quarter, which earns interest at 0.25% per month. What is his capital after 1.5 years?

$R_N =$?

with $r = \$300 \quad m_r = 4 \quad m_z = 12 \quad n = 1.5$

$$N = m_r \cdot n = 4 \cdot 1.5 = 6 \quad j = 0.0025$$

$$j^* = (1+j)^{m_z/m_r} - 1 =$$

$$= (1.0025)^{12/4} - 1 = 0.007519$$

$$R_N = r \cdot \frac{(1+j^*)^N - 1}{j^*}$$

$$R_6 = 300 \cdot \frac{(1.007519)^6 - 1}{0.007519} = \$1,834.18$$

The following table illustrates the development of the capital for the example above.

Quarter	Capital at the beginning of the quarter	Interest	Annuity	Capital at the end of the quarter
1	0.00	0.00	300	300.00
2	300.00	$300.00 \cdot$ $\cdot\, 0.007519$ $= 2.26$	300	602.26
3	602.26	$602.26 \cdot$ $\cdot\, 0.007519$ $= 4.53$	300	906.79
4	906.79	$906.79 \cdot$ $\cdot\, 0.007519$ $= 6.82$	300	$1,213.61$
5	$1,213.61$	$1,213.61 \cdot$ $\cdot\, 0.007519$ $= 9.13$	300	$1,522.74$
6	$1,522.74$	$1,522.74 \cdot$ $\cdot\, 0.007519$ $= 11.45$	300	$1,834.19$[11]

---

[11] Rounding error above 1 cent.

Present Value of Annuity $$R_0 = r \cdot \frac{(1+j^*)^N - 1}{j^* \cdot (1+j^*)^N} = r \cdot \frac{\left(1+\frac{i^*}{m_r}\right)^N - 1}{\frac{i^*}{m_r} \cdot \left(1+\frac{i^*}{m_r}\right)^N}$$

with $$j^* = \frac{i^*}{m_r}$$

<u>Example:</u>

Eva wins an amount of money that earns interest at 0.2% per month at her commercial bank. She wants to be paid $6,000 at the end of each half-year for two years, so that this amount of money will be used up at the end of the 4th semester, i.e., after two years. What is the amount of money she has won?

$R_0 =$?

with $r = \$6{,}000 \quad m_r = 2 \quad m_z = 12 \quad n = 2$

$N = m_r \cdot n = 2 \cdot 2 = 4 \qquad j = 0.002$

$$j^* = (1+j)^{m_z/m_r} - 1 =$$

$$= (1.002)^{12/2} - 1 = 0.01206$$

$$R_0 = r \cdot \frac{(1+j^*)^N - 1}{j^* \cdot (1+j^*)^N}$$

$$= 6{,}000 \cdot \frac{(1.01206)^4 - 1}{0.01206 \cdot (1.01206)^4} = \$23{,}293.49$$

The following table illustrates the development of the capital for the example above.

Half-year	Capital at the beginning of the half-year	Interest	Annuity	Capital at the end of the half-year
1	$23{,}293.49$	$23{,}293.49 \cdot 0.01206 = 280.92$	$6{,}000$	$23{,}293.49$ $+280.92$ $\underline{-6{,}000.00}$ $= 17{,}574.41$
2	$17{,}574.41$	$17{,}574.41 \cdot 0.01206 = = 211.95$	$6{,}000$	$17{,}574.41$ $+211.95$ $\underline{-6{,}000.00}$ $= 11{,}786.36$
3	$11{,}786.36$	$11{,}786.36 \cdot 0.01206 = 142.14$	$6{,}000$	$11{,}786.36$ $+142.14$ $\underline{-6{,}000.00}$ $= 5{,}928.50$
4	$5{,}928.50$	$5{,}928.50 \cdot 0.01206 = 71.50$	$6{,}000$	$5{,}928.50$ $+71.50$ $\underline{-6{,}000.00}$ $= 0.00$

Annuity payment in advance

Amount of Annuity

$$R_N = r \cdot (1+j^*) \cdot \frac{(1+j^*)^N - 1}{j^*}$$

$$= r \cdot \frac{(1+j^*)^{N+1} - (1+j^*)}{j^*}$$

or

$$R_N = r \cdot \left(1 + \frac{i^*}{m_r}\right) \cdot \frac{\left(1+\frac{i^*}{m_r}\right)^N - 1}{\frac{i^*}{m_r}}$$

$$= r \cdot \frac{\left(1+\frac{i^*}{m_r}\right)^{N+1} - \left(1+\frac{i^*}{m_r}\right)}{\frac{i^*}{m_r}}$$

with $j^* = \dfrac{i^*}{m_r}$

Example:

At the beginning of each quarter, Paul deposits an amount of \$300 into a savings account, which earns interest at a rate of 0.25 % per month. What is his capital after 1.5 years?

$R_N =$?

with $r = \$300 \quad m_r = 4 \quad m_z = 12 \quad n = 1.5$

$$N = m_r \cdot n = 4 \cdot 1.5 = 6 \qquad j = 0.0025$$

$$j^* = (1+j)^{m_z/m_r} - 1 =$$

$$= (1.0025)^{12/4} - 1 = 0.007519$$

$$R_N = r \cdot \frac{(1+j^*)^{N+1} - (1+j)^*}{j^*}$$

$$R_6 = 300 \cdot \frac{(1.007519)^{6+1} - (1.007519)}{0.007519}$$

$$= \$1,847.97$$

$$= R_6^{\text{in arrears}} \cdot 1.007519 = \$1,834.18 \cdot 1.007519$$

Present Value of Annuity

$$R_0 = r \cdot (1+j^*) \cdot \frac{(1+j^*)^N - 1}{j^* \cdot (1+j^*)^N} =$$

$$= r \cdot \frac{(1+j^*)^{N+1} - (1+j^*)}{j^* \cdot (1+j^*)^N} =$$

$$= r \cdot \frac{(1+j^*)^N - 1}{j^* \cdot (1+j^*)^{N-1}}$$

or

$$R_0 = r \cdot \left(1 + \frac{i^*}{m_r}\right) \cdot \frac{\left(1 + \frac{i^*}{m_r}\right)^N - 1}{\frac{i^*}{m_r} \cdot \left(1 + \frac{i^*}{m_r}\right)^N} =$$

$$= r \cdot \frac{\left(1 + \frac{i^*}{m_r}\right)^{N+1} - \left(1 + \frac{i^*}{m_r}\right)}{\frac{i^*}{m_r} \cdot \left(1 + \frac{i^*}{m_r}\right)^N} =$$

$$= r \cdot \frac{\left(1+\frac{i^*}{m_r}\right)^N - 1}{\frac{i^*}{m_r} \cdot \left(1+\frac{i^*}{m_r}\right)^{N-1}}$$

with $\quad j^* = \dfrac{i^*}{m_r}$

Example:

Maria wins an amount of money that earns her interest at her commercial bank at 0.2 % per month. She wants to be paid $6,000 at the beginning of each half-year for two years, so this amount of money will be used up by the beginning of the fourth semester. What is the amount of money she has won?

$R_0 =$?

with $r = \$6,000 \quad m_r = 2 \quad m_z = 12 \quad n = 2$

$N = m_r \cdot n = 2 \cdot 2 = 4 \quad j = 0.002$

$j^* = (1+j)^{m_z/m_r} - 1 = (1.002)^{12/2} - 1 = 0.01206$

$$R_0 = r \cdot \frac{(1+j^*)^N - 1}{j^* \cdot (1+j^*)^{N-1}} =$$

$$= 6,000 \cdot \frac{(1.01206)^4 - 1}{0.01206 \cdot (1.01206)^3} = \$23,574.41$$

$$= R_0^{\text{in arrears}} \cdot 1.01206 = \$23,293.49 \cdot 1.01206$$

(c) Annuity period > interest period

Annuity payment in arrears

Amount of Annuity

$$R_N = r \cdot \left(m_r + \frac{j}{2} \cdot (m_r - 1) \right) \cdot \frac{(1+j)^N - 1}{j}$$

with $j =$ relative interest rate

$$j = \frac{1}{m_z}$$

and $N = m_z \cdot n$

$m_z =$ number of sub-annual interest periods $=$

$=$ number of interest periods p.a.

$m_r =$ number of annuity periods per interest periods

Example:

Nawid deposits an amount of \$300 into a savings account at the end of each month, which earns interest at 0.25 % per quarter. What is his capital after 1.5 years?

$R_N =$?

with $r = \$300$

$m_r =$ number of annuity periods per interest periods $= 3$

$m_z =$ number of interest periods p.a. $= 4$

$N = m_z \cdot n = 4 \cdot 1.5 = 6$

mit $n = 1.5$

$$j = 0.0025$$

$$R_N = r \cdot \left(m_r + \frac{j}{2} \cdot (m_r - 1) \right) \cdot \frac{(1+j)^N - 1}{j}$$

$$R_6 = 300 \cdot \left(3 + \frac{0.0025}{2} \cdot (3-1) \right) \cdot \frac{1.0025^6 - 1}{0.0025} = \$5,438.39$$

Alternative Calculation Using the ICMA Method

In financial practice, it may be required by law to use a specific method for a mixed interest rate (see chapter 7.1.2.3). As an example, the ICMA method will be followed in the following.[12]

The relative interest rate j is then to be adjusted to the annuity period as follows:

$$(1+j^*)^{m_r} = (1+j)^3 = 1.0025$$

with $j^* =$ the relative interest rate emancipated over $m_r \cdot m_z \cdot n$ periods

$$\Rightarrow q^* = (1+j^*) = 1.0025^{1/3} = 1.00083264$$

$$\Rightarrow j^* = 0.00083264 = 0.83264\,‰ \text{ monthly}$$

$$R_N = r \cdot \frac{(q^*)^{m_r \cdot m_z \cdot n} - 1}{j^*}$$

$$R_6 = 300 \cdot \frac{(1.00083264)^{3 \cdot 4 \cdot 1.5} - 1}{0.00083264} = \$5,438.39$$

---

[12] For Germany according to §16 PAngV.

Present Value of Annuity $R_0 = r \cdot \left(m_r + \frac{j}{2} \cdot (m_r - 1) \right) \cdot \frac{(1+j)^N - 1}{j \cdot (1+j)^N}$

with $j =$ relative interest rate

$$j = \frac{1}{m_z}$$

and $N = m_z \cdot n$

$m_z =$ number of sub-annual interest periods =

$=$ number of interest periods p.a.

$m_r =$ number of annuity periods per interest periods

Example:

Eva wins an amount of money that earns interest at her commercial bank at 1.2% per half-year, i.e. per semester. She would like to be paid out \$50 at the end of each month for two years, so that this amount of money will be used up at the end of the 4th semester, i.e., after two years. What is the amount of money she has won?

$R_0 =$?

with $r = \$50$

$m_r =$ number of annuity periods per interest periods $= 6$

$m_z =$ number of interest periods p.a. $= 2$

$N = m_z \cdot n = 2 \cdot 2 = 4$

with $n = 2$

$j = 0.012$

$$R_0 = r \cdot \left(m_r + \frac{j}{2} \cdot (m_r - 1)\right) \cdot \frac{(1+j)^N - 1}{j \cdot (1+j)^N} =$$

$$= 50 \cdot \left(6 + \frac{0.012}{2} \cdot (6-1)\right) \cdot \frac{1.012^4 - 1}{0.012 \cdot 1.012^4} = \$1,170.67$$

Alternative Calculation Using the ICMA Method

Cf. chapter 7.1.2.3

The relative interest rate j is then to be adjusted to the annuity period as follows:

$$(1+j^*)^{m_r} = (1+j)^6 = 1.012$$

with $j^* =$ the relative interest rate emancipated over $m_r \cdot m_z \cdot n$ periods

$\Rightarrow q^* = (1+j^*) = 1.012^{1/6} = 1.001990073$

$\Rightarrow j^* = 0.001990073 = 1.990073‰$ monthly

$$R_0 = r \cdot \frac{(q^*)^{m_r \cdot m_z \cdot n} - 1}{j^* \cdot (q^*)^{m_r \cdot m_z \cdot n}}$$

$$R_0 = 50 \cdot \frac{(1.001990073)^{6 \cdot 2 \cdot 2} - 1}{0.001990073 \cdot (1.001990073)^{6 \cdot 2 \cdot 2}} = 1,170.66$$ [13]

---

[13] The difference from the original calculation using the formula $R_0 = r \cdot \left(m_r + \frac{j}{2} \cdot (m_r - 1)\right) \cdot \frac{(1+j)^N - 1}{j \cdot (1+j)^N}$ is 1 cent.

Annuity payment in advance

Amount of Annuity

$$R_N = r \cdot \left(m_r + \frac{j}{2} \cdot (m_r + 1) \right) \cdot \frac{(1+j)^N - 1}{j}$$

with $j =$ relative interest rate

$$j = \frac{1}{m_z}$$

and $N = m_z \cdot n$

$m_z =$ number of sub-annual interest periods =

$=$ number of interest periods p.a.

$m_r =$ number of annuity periods per interest periods

Example:

Paul deposits an amount of $100 into a savings account at the beginning of each month, which earns interest at 0.75 % each quarter. What is his capital after 1.5 years?

$R_N =$?

with $r = \$100$

$m_r =$ number of annuity periods per interest periods $= 3$

$m_z =$ number of interest periods p.a. $= 4$

$N = m_z \cdot n = 4 \cdot 1.5 = 6$

with $n = 1.5$

$j = 0.0075$

$$R_N = r \cdot \left(m_r + \frac{j}{2} \cdot (m_r + 1) \right) \cdot \frac{(1+j)^N - 1}{j}$$

$$R_6 = 100 \cdot \left(3 + \frac{0.0075}{2} \cdot (3+1) \right) \cdot \frac{1.0075^6 - 1}{0.0075} = \$1,843.26$$

Alternative Calculation Using the ICMA Method

Cf. chapter 7.1.2.3

The relative interest rate j is then to be adjusted to the annuity period as follows:

$$(1+j^*)^{m_r} = (1+j)^3 = 1.0075$$

with $j^* =$ the relative interest rate emancipated over $m_r \cdot m_z \cdot n$ periods

$\Rightarrow q^* = (1+j^*) = 1.0075^{1/3} = 1.002493776$

$\Rightarrow j^* = 0.002493776 = 2,493776‰$ monthly

$$R_N = r \cdot \frac{q^* \cdot [(q^*)^{m_r \cdot m_z \cdot n} - 1]}{j^*}$$

$$R_6 = 100 \cdot \frac{1.002493776 \cdot \left[(1.002493776)^{3 \cdot 4 \cdot 1.5} - 1\right]}{0.002493776} = \$1,843.25$$ [14]

Present Value of Annuity $R_0 = r \cdot \left(m_r + \frac{j}{2} \cdot (m_r + 1)\right) \cdot \frac{(1+j)^N - 1}{j \cdot (1+j)^N}$

with $j =$ relative interest rate

$$j = \frac{1}{m_z}$$

and $N = m_z \cdot n$

$m_z =$ number of sub-annual interest periods$=$

$=$ number of interest periods p.a.

$m_r =$ number of annuity periods per interest periods

Example:

Maria wins an amount of money that earns her interest at her commercial bank at 0.6 % quarterly. She wants to be paid $50 at the beginning of each month for two years, so that this amount of money will be used up at the beginning of the 24th month. What is the amount of money she has won?

---

[14] The difference from the original calculation using the formula $R_N = r \cdot \left(m_r + \frac{j}{2} \cdot (m_r + 1)\right) \cdot \frac{(1+j)^N - 1}{j}$ is 1 cent.

$R_0 =$?

with $r = \$50$

$m_r =$ number of annuity periods per interest periods $= 3$

$m_z =$ number of interest periods p.a. $= 4$

$N = m_z \cdot n = 4 \cdot 2 = 8$

mit $n = 2$

$j = 0.006$

$$R_0 = r \cdot \left(m_r + \frac{j}{2} \cdot (m_r + 1) \right) \cdot \frac{(1+j)^N - 1}{j \cdot (1+j)^N} =$$

$$= 50 \cdot \left(3 + \frac{0.006}{2} \cdot (3+1) \right) \cdot \frac{1.006^8 - 1}{0.006 \cdot 1.006^8} = \$1,172.91$$

Alternative Calculation Using the ICMA Method

Cf. chapter 7.1.2.3

The relative interest rate j is then to be adjusted to the annuity period as follows:

$$(1+j^*)^{m_r} = (1+j)^3 = 1.006$$

with $j^* =$ the relative interest rate emancipated over $m_r \cdot m_z \cdot n$ periods

$\Rightarrow q^* = (1+j^*) = 1.006^{1/3} = 1.001996013$

$\Rightarrow j^* = 0.001996013 = 1.996013‰$ monthly

$$R_0 = r \cdot \frac{q^* \cdot [(q^*)^{m_r \cdot m_z \cdot n} - 1]}{j^* \cdot (q^*)^{m_r \cdot m_z \cdot n}}$$

$$R_0 = 50 \cdot \frac{1.001996013 \cdot \left[(1.001996013)^{3 \cdot 4 \cdot 2} - 1\right]}{0.001996013 \cdot (1.001996013)^{3 \cdot 4 \cdot 2}} = \$1,172.91$$

7.4.3 Finite, Variable Annuity

These are recurring payments which are made at regular intervals and the extent of which changes over the course of time. The amount of the annuity changes over time.

Changes of annuity can

- vary without system (irregular annuity)
- be systematic (regular annuity)
  - arithmetic progressive annuity
  - geometric progressive annuity

7.4.3.1 Irregular Annuity

Amount of Annuity

$$R_n = q^n \cdot \sum_{k=1}^{n} r_k \cdot q^{-k}$$

Present Value of Annuity

$$R_0 = \sum_{k=1}^{n} r_k \cdot q^{-k}$$

Interest Rate
(R_n given)

$$f(i) = -R_n + q^n \cdot \sum_{k=1}^{n} r_k \cdot q^{-k}$$

$$f'(i) = q^{n-1} \cdot \sum_{k=1}^{n} (n-k) \cdot r_k \cdot q^{-k}$$

Interest Rate
(R_0 given)

$$f(i) = -R_0 + \sum_{k=1}^{n} r_k \cdot q^{-k}$$

$$f'(i) = -q^{-1} \cdot \sum_{k=1}^{n} k \cdot r_k \cdot q^{-k}$$

Example 1:

Camilla pays irregular amounts into her savings account at the beginning of each year for three years:

1th year:	$\$1,000$
2nd year:	$\$2,500$
3rd year:	$\$3,200$

The credit balance earns interest at 2 % p.a.

What amount of money does Camilla have after three years?

$r_1 = \$1,000 \quad r_2 = \$2,500 \quad r_3 = \$3,200$

$n = 3$ years $\quad q = 1.02 \quad k = 3$

$$R_0 = \$1,000 \cdot 1.02^{-1} + \$2,500 \cdot 1.02^{-2} + \$3,200 \cdot 1.02^{-3} = \$6,398.74$$

$$R_n = R_3 = 6,398.74 \cdot 1.02^3 = \$6,790.39$$

After three years, Camilla has $6,790.39.

Example 2:

The conditions from example 1 (see above) apply. That means Camilla pays the following amounts at the beginning of each year over three years:

$r_1 = \$1{,}000 \quad r_2 = \$2{,}500 \quad r_3 = \$3{,}200$

$n = 3$ years

$R_0 = \$6{,}398.74$

$R_n = R_3 = \$6{,}790.39$

The interest rate i is not known and is to be determined approximately using Newton's method of approximation. The market suggests that this interest rate is likely to be between 1 % and 4 %. Therefore, $i = 0.03$ is to be assumed as a fictitious initial value.

Newton's Method of approximation[15] is to be applied with

$$i_{k+1} = i_k - \frac{f(i_k)}{f'(i_k)}$$

i_k corresponds to the (fictitious) initial value $=$ first (fictitiously assumed) approximate value

i_{k+1} corresponds to the next (calculated) approximate value

---

[15] See chapter 4.5.2.

a) at a given amount of annuity R_n

$$f(i) = -R_n + r \cdot \frac{q^n - 1}{i} + \frac{d}{i} \cdot \left(\frac{q^n - 1}{i} - n \right)$$

$$f'(i) = \frac{r}{i} \cdot \left(nq^{n-1} - \frac{q^n - 1}{i} \right) + \frac{d}{i^2} \cdot \left(n + nq^{n-1} - 2\frac{q^n - 1}{i} \right)$$

b) at a given present value of annuity R_0

$$f(i) = -R_0 + r \cdot \frac{q^n - 1}{i \cdot q^n} + \frac{d}{i} \cdot \left(\frac{q^n - 1}{i \cdot q^n} - n \cdot q^{-n} \right)$$

$$f'(i) = \frac{r}{i} \cdot \left(\frac{n}{q^{n+1}} - \frac{q^n - 1}{iq^n} \right) + \frac{d}{i^2} \cdot \left(\frac{n}{q^{n+1}}(1 + q + in) - 2\frac{q^n - 1}{iq^n} \right)$$

$i_k = 0.03$ fictitiously assumed initial value within the interval

$[1\,\%;\ 4\,\%]$

$i_{k+1} =$?

1. Deterine approximate value

   $i = 0.03$ (estimated) $q = 1.03$ $R_n = \$6,790.39$

   determine: $f(0.03)$ determine: $f'(0.03)$

$\sum_{k=1}^{n} r_k \cdot q^{-k}$

$k = 1:\ 1{,}000 \cdot 1.03^{-1} \quad = 970.874$

$k = 2:\ 2{,}500 \cdot 1.03^{-2} \quad = 2{,}356.49$

$k = 3:\ 3{,}200 \cdot 1.03^{-3} \quad \underline{= 2{,}928.45}$

$\sum 6{,}255.81$

$\sum_{k=1}^{n} (n-k) \cdot r_k \cdot q^{-k}$

$k = 1:\ (3-1) \cdot 1{,}000 \cdot 1.03^{-1} \ = 1{,}941.75$

$k = 2:\ (3-2) \cdot 2{,}500 \cdot 1.03^{-2} \ = 2{,}356.49$

$k = 3:\ (3-3) \cdot 3{,}200 \cdot 1.03^{-3} \ \underline{= 0}$

$\sum 4{,}298.24$

$1.03^3 \cdot 6{,}255.81 = 6{,}835.89$

$f(0.03) = -6{,}790.39 + 6{,}835.89 =$

$= 45.5$

$f'(0.03) = 1.03^{3-1} \cdot 4{,}298.24 =$

$= 4{,}560$

$$i_{n+1} = 0.03 - \frac{45.5}{4{,}560} = 0.020022$$

$i_1 = 0.020022 =$ first approximate value $=$ new initial value

2. Determine approximate value

$i_1 = 0.020022 \quad q = 1.020022 \quad R_n = 6{,}790.39$

determine: $f(0.020022)$

$\sum_{k=1}^{n} r_k \cdot q^{-k}$

determine: $f'(0.020022)$

$\sum_{k=1}^{n} (n-k) \cdot r_k \cdot q^{-k}$

$k = 1:\ 1{,}000 \cdot 1.020022^{-1} = 980.371$ $\quad k = 1:\ (3-1) \cdot 1{,}000 \cdot 1.020022^{-1} = 1{,}960.74$

$k = 2:\ 2{,}500 \cdot 1.020022^{-2} = 2{,}402.82$ $\quad k = 2:\ (3-2) \cdot 2{,}500 \cdot 1.020022^{-2} = 2{,}402.82$

$k = 3:\ 3{,}200 \cdot 1.020022^{-3} \underline{= 3{,}015.24}$ $\quad k = 3:\ (3-3) \cdot 3{,}200 \cdot 1.020022^{-3} \underline{= 0}$

$\sum 6{,}398.43$ $\quad \sum 4{,}363.56$

$1.020022^3 \cdot 6{,}398.43 = 6{,}790.5$

$f(0.020022) = -6{,}790.39 + 6{,}790.5 = = 0.11$ $\quad f'(0.020022) = 1.020022^{3-1} \cdot 4{,}363.56 = = 4{,}540.04$

$$i_{n+1} = 0.020022 - \frac{0.11}{4{,}540.04} = 0.019998$$

$$i_2 = 0.019998$$

3. Determine approximate value

$$i_2 = 0.019998 \quad q = 1.019998 \quad R_n = 6{,}790.39$$

determine: $f(0.019998)$ $\quad$ determine: $f'(0.019998)$

$\sum\limits_{k=1}^{n} r_k \cdot q^{-k}$ $\quad \sum\limits_{k=1}^{n} (n-k) \cdot r_k \cdot q^{-k}$

$k = 1:\ 1{,}000 \cdot 1.019998^{-1} = 980.384$ $\quad k = 1:\ (3-1) \cdot 1{,}000 \cdot 1.019998^{-1} = 1{,}960.79$

$k = 2:\ 2{,}500 \cdot 1.019998^{-2} \ = 2{,}402.93 \quad k = 2:\ (3-2) \cdot 2{,}500 \cdot 1.019998^{-2} \ = 2{,}402.93$

$k = 3:\ 3{,}200 \cdot 1.019998^{-3} \ \underline{= 3{,}015.45} \quad k = 3:\ (3-3) \cdot 3{,}200 \cdot 1.019998^{-3} \ \underline{= 0}$

$\sum 6{,}398.77 \qquad \sum 4{,}363.72$

$1.019998^3 \cdot 6{,}398.77 = 6{,}790.39$

$f(0.019998) = -6{,}790.39 + 6{,}790.39 = \quad f'(0.019998) = 1.019998^{3-1} \cdot 4{,}363.72 =$

$= 0 \qquad = 4{,}540$

$$i_{n+1} = 0.019998 - \frac{0}{4{,}540} = 0.019998$$

$i_3 = 0.019998$

Solution

By (iteratively) continuing the determination of further approximate values, the following is obtained:

$i_1 = 0.020022$

$i_2 = 0.019998$ ($\approx 0.02 \rightarrow$ rounded to two decimal places)

$i_3 = 0.019998$ ($\approx 0{,}02 \rightarrow$ rounded to two decimal places)

→ The result for the interest rate i is 0.019998, i.e. approximately $i = 0.02$ or $p = 2\,\%$

7.4.3.2 Arithmetic Progressive Annuity

The annuity payment increases from period to period by a predetermined amount.

r	equivalent to the annuity payment at point in time $k = 1$
d	equivalent to the difference between two successive annuity payments $d = r_{k+1} - r_k$

Amount of Annuity

$$r_1 = r$$

$$r_2 = r + d$$

$$r_3 = r + 2d$$

$$\vdots$$

$$r_n = r + (n-1) \cdot d$$

$$R_n = \sum_{k=1}^{n} \Big[r + (k-1) \cdot d \Big] \cdot q^{n-k}$$

$$R_n = r \cdot \frac{q^n - 1}{i} + \frac{d}{i} \cdot \left(\frac{q^n - 1}{i} - n \right)$$

Present Value of Annuity

$$R_0 = r \cdot \frac{q^n - 1}{i \cdot q^n} + \frac{d}{i} \cdot \left(\frac{q^n - 1}{i \cdot q^n} - n \cdot q^{-n} \right)$$

Annuity (R_n given)

$$r = \frac{R_n \cdot i + d \cdot n}{q^n - 1} - \frac{d}{i}$$

Annuity (R_0 given) $$r = \frac{R_0 \cdot i \cdot q^n + d \cdot n}{q^n - 1} - \frac{d}{i}$$

Interest Rate (R_n given) $$f(i) = -R_n + r \cdot \frac{q^n - 1}{i} + \frac{d}{i} \cdot \left(\frac{q^n - 1}{i} - n \right)$$

Interest Rate (R_0 given) $$f(i) = -R_0 + r \cdot \frac{q^n - 1}{i \cdot q^n} + \frac{d}{i} \cdot \left(\frac{q^n - 1}{i \cdot q^n} - n \cdot q^{-n} \right)$$

Period (R_n given) $$f(n) = -R_n + r \cdot \frac{q^n - 1}{i} + \frac{d}{i} \cdot \left(\frac{q^n - 1}{i} - n \right)$$

Period (R_0 given) $$f(n) = -R_0 + r \cdot \frac{q^n - 1}{i \cdot q^n} + \frac{d}{i} \cdot \left(\frac{q^n - 1}{i \cdot q^n} - n \cdot q^{-n} \right)$$

Example 1:

Mrs. Penny would like to save money for her newborn godchild, Camilla, which she will receive on her 18th birthday. She starts to deposit $500 in the first year, but would like to increase the amount by $50 each year. The interest rate is $0.7\,\%$ p.a.

How much money will Camilla receive on her 18th birthday?

$$R_n = \$500 \cdot \frac{1.007^{18} - 1}{0.007} + \frac{\$50}{0.007} \cdot \left(\frac{1.007^{18} - 1}{0.007} - 18 \right)$$

$$R_n \approx \$17,499.27$$

Camilla is entitled to $17,499.27 at the end of the 18 years, if Mrs. Penny - as intended - deposits the increasing amount into the account each year.

Example 2:

Steven receives an annuity (= annuity payment) over 8 years. The first annuity payment is \$2,000, which is increased by \$100 per year. The interest rate is 5 % p.a.

What is the present value of this annuity?

$$n = 8 \quad i = 0.05 \quad q = 1.05 \quad r = \$2,000 \quad d = \$100$$

$$R_0 = 2,000 \cdot \frac{1.05^8 - 1}{0.05 \cdot 1.05^8} + \frac{100}{0.05} \cdot \left(\frac{1.05^8 - 1}{0.05 \cdot 1.05^8} - 8 \cdot 1.05^{-8} \right)$$

$$= \$15,023.42$$

The present value of this arithmetic progressed annuity is \$15,023.42.

Example 3:

Mrs. Pfennig saved a total of \$9,736.46 (R_n) for her grandson for 18 years. The money earned interest at 0.5 % p.a. and she increased the amount by \$50 annually.

How much was the first annuity payment of this annuity?

$$R_n = \$9,736.46 \quad i = 0.005 \quad d = \$50 \quad n = 18 \quad q = 1.005$$

$$r = \frac{9,736.46 \cdot 0.005 + 50 \cdot 18}{1.005^{18} - 1} - \frac{50}{0.005}$$

$$r = \$100$$

The first annuity payment was \$100.

Example 4:

Given the present value of annuity of \$15,023.42, the annuity payment was increased by \$100 per year for eight years at an interest rate of 5 % p.a.

How much was the first annuity payment of this annuity?

$R_0 = \$15{,}023.42 \quad i = 0.05 \quad q = 1.05 \quad d = \$100 \quad n = 8$

$$r = \frac{15{,}023.42 \cdot 0.05 \cdot 1.05^8 + 100 \cdot 8}{1.05^8 - 1} - \frac{100}{0.05}$$

$r = 1{,}999.9998 \approx \$2{,}000$

The first annuity payment was \$2,000.

Calculation of the Interest Rate i

Newton's Method of approximation (see chapter 4.5.2) is to applied with

$$i_{k+1} = i_k - \frac{f(i_k)}{f'(i_k)}$$

i_k	corresponds to the (fictitious) initial value =first (fictitiously assumed) approximate value
i_{k+1}	corresponds to the next (calculated) approximate value

a) at a given amount of annuity R_n

$$f(i) = -R_n + r \cdot \frac{q^n - 1}{i} + \frac{d}{i} \cdot \left(\frac{q^n - 1}{i} - n \right)$$

$$f'(i) = \frac{r}{i} \cdot \left(nq^{n-1} - \frac{q^n - 1}{i} \right) + \frac{d}{i^2} \cdot \left(n + nq^{n-1} - 2\frac{q^n - 1}{i} \right)$$

b) at a given present value of annuity R_0

$$f(i) = -R_0 + r \cdot \frac{q^n - 1}{i \cdot q^n} + \frac{d}{i} \cdot \left(\frac{q^n - 1}{i \cdot q^n} - n \cdot q^{-n} \right)$$

$$f'(i) = \frac{r}{i} \cdot \left(\frac{n}{q^{n+1}} - \frac{q^n - 1}{iq^n} \right) + \frac{d}{i^2} \cdot \left(\frac{n}{q^{n+1}}(1 + q + in) - 2\frac{q^n - 1}{iq^n} \right)$$

Calculation of the Period n

Newton’s Method of approximation (see chapter 4.5.2) is to applied with

$$n_{k+1} = n_k - \frac{f(n_k)}{f'(n_k)}$$

n_k corresponds to the (fictitious) initial value =first (fictitiously assumed) approximate value

n_{k+1} corresponds to the next (calculated) approximate value

a) at a given amount of annuity R_n

$$f(n) = -R_n + r \cdot \frac{q^n - 1}{i} + \frac{d}{i} \cdot \left(\frac{q^n - 1}{i} - n \right)$$

$$f'(n) = -\frac{d}{i} + \frac{q^n lnq}{i} \cdot \left(r + \frac{d}{i} \right)$$

b) at a given present value of annuity R_0

$$f(n) = -R_0 + r \cdot \frac{q^n - 1}{i \cdot q^n} + \frac{d}{i} \cdot \left(\frac{q^n - 1}{i \cdot q^n} - n \cdot q^{-n} \right)$$

$$f'(n) = -q^{-n} \left(\frac{d}{i} - \frac{lnq}{i} \cdot \left(r + nd + \frac{d}{i} \right) \right)$$

Example 1:

Mrs. Pfennig wants to save money for her just-born godchild, Penny, which she will receive on her 18th birthday. She starts paying in \$500 in the first year, but would like to increase the amount by \$50 each year.

The interest rate is 0.7% p.a. On her 18th birthday, i.e., after 18 years of life, Penny should have R_n = $17,499.27 (see example above) at her disposal.

$r = \$500$

$d = \$50$

$i = 0.007\ (0.7\%) \qquad q = 1.007$

As a fictitious (estimated) initial value n_0 = 20 are to be assumed with q^n = 1.007^{20} ≈ 1.1497

$$n_{k+1} = n_k - \frac{f(n)}{f'(n)}$$

$$f(n) = -R_n + r \cdot \frac{q^n - 1}{i} + \frac{d}{i} \cdot \left(\frac{q^n - 1}{i} - n \right)$$

$$f(20) = -17,499.27 + 500 \cdot \frac{1.00^{20} - 1}{0.007} + \frac{50}{0.007} \cdot \left(\frac{1.007^{20} - 1}{0.007} - 20 \right)$$

$$f(20) = 3,105.65$$

$$f'(n) = -\frac{d}{i} + \frac{q^n \cdot ln(q)}{i} \cdot \left(r + \frac{d}{i} \right)$$

$$f'(20) = -\frac{50}{0.007} + \frac{1.007^{20} \cdot ln(1.007)}{0.007} \cdot \left(500 + \frac{50}{0.007} \right)$$

$$f'(20) = 1,613.62$$

$$n_{k+1} = n_k - \frac{f(x)}{f'(x)}$$

$$n_0 = 20$$

$$n_1 = 20 - \frac{3,105.65}{1,613.62}$$

$n_1 = 18.075352$ = first approximate value = new initial value

The approximate value $n_1 = 18.075352$ is now inserted in place of the initially fictitiously assumed initial value of $n_0 = 20$ into the following formula

$$n_{k+1} = n_k - \frac{f(n_k)}{f'(n_k)}$$

This iterative procedure is continued until the approximate values change only marginally and converge to a fixed value. In the example above, the following values are obtained:

$$n_2 = 18.000114$$

$$n_3 = 18.000000$$

$$n_4 = 18.000000$$

Thus, the result is n = 18 years.

Example 2:

Michelle deposited money into her account each month. In the first month she deposited \$50, in the following months she increased her deposit by \$25 each. The balance earned exponential interest at 1 % per month. Now, at the end of the period of this savings plan, Michelle gets R_n =\$1,678.64 paid out.

Over how many months n did this savings plan run?

The exact period n is no longer known to Michelle and is to be determined approximately using Newton's method of approximation. Michelle vaguely remembers that the period of this savings plan was between 8 and 13 months. Therefore, one year, i.e. 12 months, is to be assumed as a fictitious initial value.

$$n_{k+1} = n_k - \frac{f(n)}{f'(n)}$$

$$f(n) = -R_n + r \cdot \frac{q^n - 1}{i} + \frac{d}{i} \cdot \left(\frac{q^n - 1}{i} - n \right)$$

$$f(12) = -1,678.64 + 50 \cdot \frac{1.01^{12} - 1}{0.01} + \frac{25}{0.01} \cdot \left(\frac{1.01^{12} - 1}{0.01} - 12 \right) =$$

$$= \$661.734$$

$$f'(n) = -\frac{d}{i} + \frac{q^n \cdot ln(q)}{i} \cdot \left(r + \frac{d}{i} \right)$$

$$f'(12) = -\frac{25}{0.01} + \frac{1.01^{12} \cdot ln(1.01)}{0.01} \cdot \left(50 + \frac{25}{0.01} \right) =$$

$$= \$359.132$$

$$n_{k+1} = n_k - \frac{f(x)}{f'(x)}$$

$$n_0 = 12$$

$$n_1 = 12 - \frac{661.734}{359.132}$$

$n_1 = 10.1574 =$ first approximate value = new initial value

The approximate value $n_1 = 10.1574$ is now inserted in place of the initially fictitiously assumed initial value of $n_0 = 12$ into the following formula

$$n_{k+1} = n_k - \frac{f(n_k)}{f'(n_k)}$$

This iterative procedure is continued until the approximate values change only marginally and converge to a fixed value. In the example above, the following calculations and values are obtained:

2. Determine approximate value

$$f(10.1574) = -1{,}678.64 + 50 \cdot \frac{1.01^{10.1574} - 1}{0.01} + \frac{25}{0.01} \cdot$$

$$\cdot \left(\frac{1.01^{10.1574} - 1}{0.01} - 10.1574 \right) = 48.0077$$

$$f'(10.1574) = -\frac{25}{0.01} + \frac{1.01^{10.1574} \cdot ln 1.01}{0.01} \cdot \left(50 + \frac{25}{0.01} \right) = 307.185$$

$$n_{k+1} = 10.1574 - \frac{48.0077}{307.185}$$

$$n_2 = 10.0011$$

3. Determine approximate value

$$f(10.0011) = -1{,}678.64 + 50 \cdot \frac{1.01^{10.0011} - 1}{0.01} + \frac{25}{0.01} \cdot$$

$$\cdot \left(\frac{1.01^{10.0011} - 1}{0.01} - 10.0011 \right) = 0.335072$$

$$f'(10.0011) = -\frac{25}{0.01} + \frac{1.01^{10.0011} \cdot ln 1.01}{0.01} \cdot \left(50 + \frac{25}{0.01} \right) = 302.826$$

$$n_{k+1} = 10.0011 - \frac{0.335072}{302.826}$$

$$n_3 = 9.99999 \approx 10$$

4. Determine approximate value

$$f(9.99999) = -1,678.64 + 50 \cdot \frac{1.01^{9.99999} - 1}{0.01} + \frac{25}{0.01} \cdot$$

$$\cdot \left(\frac{1.01^{9.99999} - 1}{0.01} - 9.99999 \right) = -0.001048$$

$$f'(9.99999) = -\frac{25}{0.01} + \frac{1.01^{9.99999} \cdot ln 1.01}{0.01} \cdot \left(50 + \frac{25}{0.01} \right) = 302.795$$

$$n_{k+1} = 9.99999 - \frac{-0.001048}{302.795}$$

$$n_4 = 9.99999 \approx 10$$

5. Determine approximate value (test)

$$f(10) = -1,678.64 + 50 \cdot \frac{1.01^{10} - 1}{0.01} + \frac{25}{0.01} \cdot \left(\frac{1.01^{10} - 1}{0.01} - 10 \right) =$$

$$= 0.00198$$

$$f'(10) = -\frac{25}{0.01} + \frac{1.01^{10} \cdot ln 1.01}{0.01} \cdot \left(50 + \frac{25}{0.01} \right) = 302.796$$

$$n_{k+1} = 10 - \frac{0.00198}{302.796}$$

$$n_5 = 9.99999 \approx 10$$

By continuing the approximation the following is obtained

$n_1 = 10.1574$
$n_2 = 10.0011$

$$n_3 = 9.99999 \approx 10$$
$$n_4 = 9.99999 \approx 10$$
$$n_5 = 9.99999 \approx 10$$

The period n of this arithmetic progressive annuity was 10 months.

7.4.3.3 Geometric Progressive Annuity

The annuity payment increases annually by a given percentage.

g is equivalent to the growth factor of two successive annuity payments

$$g = \frac{r_{k+1}}{r_k}$$

(a) Interest factor $\neq$ growth factor $(q \neq g)$

Amount of Annuity	$R_n = r \cdot \frac{q^n - g^n}{q - g}$
Present Value of Annuity	$R_0 = r \cdot \frac{q^n - g^n}{(q - g) \cdot q^n}$
Annuity (R_n given)	$r = R_n \cdot \frac{q - g}{q^n - g^n}$
Annuity (R_0 given)	$r = R_0 \cdot \frac{(q - g) \cdot q^n}{q^n - g^n}$
Interest Rate (R_n given)	$f(i) = -R_n + r \cdot \frac{q^n - g^n}{q - g}$

Interest Rate
(R_0 given)

$$f(i) = -R_0 + r \cdot \frac{q^n - g^n}{(q-g) \cdot q^n}$$

Period (R_n given)

$$f(n) = -R_n + r \cdot \frac{q^n - g^n}{q-g}$$

Period (R_0 given)

$$f(n) = -R_0 + r \cdot \frac{q^n - g^n}{(q-g) \cdot q^n}$$

Example:

Mrs. Penny would like to know how the amount for her newborn godchild will change if she increases the first amount of \$500 by $10\,\%$ in the following years.

$$g = \frac{\$550}{\$500} = 1.1$$

$$R_n = \$500 \cdot \frac{1.007^{18} - 1.1^{18}}{1.007 - 1.1}$$

$$R_n \approx \$23,796.41$$

If Mrs. Penny increases the annual amount by $10\,\%$ each year, Camilla would receive \$23,796.41 on her 18th birthday.

(b) Interest factor $=$ growth factor $(q = g)$

Amount of Annuity

$$R_n = r \cdot n \cdot q^{n-1}$$

Present Value
of Annuity

$$R_0 = \frac{r \cdot n}{q}$$

Annuity (R_n given) $r = \dfrac{R_n}{n \cdot q^{n-1}}$

Annuity (R_0 given) $r = \dfrac{R_0 \cdot q}{n}$

Interest Rate (R_n given) $i = \sqrt[n-1]{\dfrac{R_n}{r \cdot n}} - 1$

Interest Rate (R_0 given) $i = \dfrac{r \cdot n}{R_0} - 1$

Period (R_n given) $f(n) = -R_n + r \cdot n \cdot q^{n-1}$

Period (R_0 given) $n = \dfrac{q \cdot R_0}{r}$

Example:

Mrs. Penny wants to save \$500 a year for Camilla again. This time, she would like to increase each annual savings amount by $5\,\%$. She is budgeting at a calculatory interest rate of $5\,\%$ p.a.

$$R_n = \$500 \cdot 18 \cdot 1.1^{18-1}$$

$$R_n \approx \$45,490.23$$

Camilla would receive \$45.490,23 on her 18th birthday.

7.4.4 Perpetuity

A perpetuity is an annuity that can be generated from the interest income of a fixed-interest investment without reducing the amount of the invested capital. Accordingly, a perpetual annuity is characterized by an infinitely long flow of payments. Consequently, the amount of annuity becomes infinitely large. Since the capital is fully preserved, the resulting return is generated "eternally".

(a) Annuity in arrears

Present Value of Annuity $$R_0 = r \cdot \frac{q^n - 1}{i \cdot q^n} \qquad \text{with } i > 0$$

$$R_0 = \frac{r}{i} \cdot \left(\frac{q^n - 1}{q^n} \right)$$

$$R_0 = \frac{r}{i} \cdot \left(\frac{q^n}{q^n} - \frac{1}{q^n} \right)$$

$$R_0 = \frac{r}{i} \cdot \left(1 - \frac{1}{q^n} \right)$$

if $q > 1$ and $i > 0$:

$$\boxed{R_0 = \lim_{n \to \infty} \frac{r}{i} \cdot \left(1 - \frac{1}{q^n} \right) = \frac{r}{i}}$$

since $\lim\limits_{n \to \infty} \frac{1}{q^n} = 0$

(b) Annuity in advance

Present Value of Annuity $$R_0 = r \cdot \frac{q \cdot (q^n - 1)}{i \cdot q^n} \qquad \text{with } i > 0$$

$$R_0 = \frac{r \cdot q}{i} \cdot \left(\frac{q^n - 1}{q^n}\right)$$

$$R_0 = \frac{r \cdot q}{i} \cdot \left(\frac{q^n}{q^n} - \frac{1}{q^n}\right)$$

$$R_0 = \frac{r \cdot q}{i} \cdot \left(1 - \frac{1}{q^n}\right)$$

if $q > 1$ and $i > 0$:

$$\boxed{R_0 = \lim_{n \to \infty} \frac{r \cdot q}{i} \cdot \left(1 - \frac{1}{q^n}\right) = \frac{r \cdot q}{i}}$$

since $\lim_{n \to \infty} \frac{1}{q^n} = 0$

Example:

Mrs. Penny owns a property that she wants to sell. She receives a net annual income of \$30,000 for renting out this property. At what price should Mrs. Penny sell the property, assuming an interest rate of $2\,\%$ p.a., if she assumes that alternatively the property would belong to her or her family eternally?

$$R_0 = \frac{\$30,000 \cdot 1.02}{0.02} = \$1,530,000$$

The “perpetual” value of the property is \$1,530,000 assuming that the interest rate is $2\,\%$ p.a. on a permanent, i.e. “perpetual” basis.

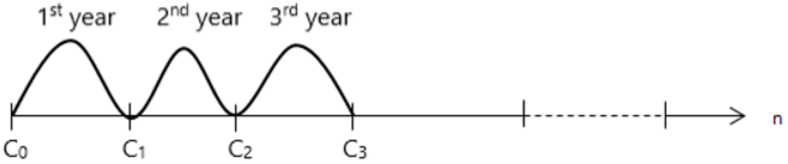
7.5 Sinking Fund Calculation

The sinking fund calculation deals with the repayment of loans, credits and mortgages. The debt is repaid in instalments within a period agreed upon in advance. The creditor expects interest to be paid.

7.5.1 Fundamental Terms

Annuity A_k	Total payment per period with $k = 1, ..., n$. Each annuity consists of an interest rate and a repayment instalment. $A_k = T_k + Z_k$
Repayment Instalment T_k	Amount that reduces the debt at the end of the period by payment, with $k = 1, ..., n$. Note: The principle amount is only reduced by the repayment instalment.
Interest Amount Z_k	Interest for the respective residual debt C_k, with $k = 1, ..., n$. To determine the interest for annual payments in arrears, the interest rate is multiplied by the residual debt of the previous year. $Z = i \cdot C_{k-1}$
Initial Debt C_0	Equivalent to the original loan amount. If n is the total number of all repayment periods, the initial debt C_0 is equal to the total of all repayment instalments T_k. $C_0 = T_1 + T_2 + ... + T_{n-1} + T_n = \sum_{k=1}^{n} T_k$

Residual Debt C_k	Equivalent to the remaining amount after k periods, with $k = 1,...,n$. The residual debt of the current period is equivalent to the difference between the residual debt amount of the previous period and the repayment instalment of the current period. $C_k = C_{k-1} - T_k$ The residual debt C_k after k periods is equal to the initial debt C_0 minus the sum of the repayment instalments T_k. $C_k = C_0 - (T_1 + T_2 + ... + T_k) = C_0 - \sum_{k=1}^{k} T_k$
Period k	Duration for which an annuity A_k is to be paid.
Repayment Period n	Total duration of the loan.

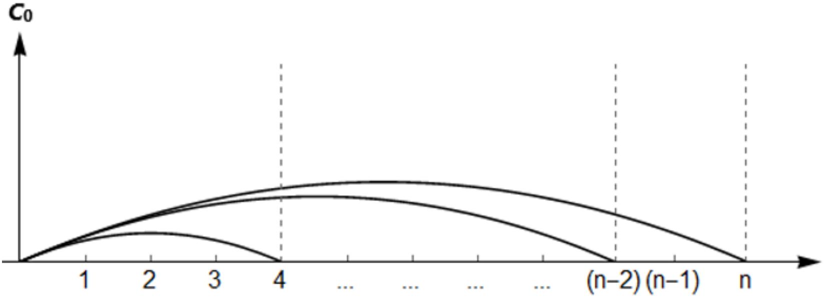


7.5.2 Annuity Repayment

With the annuity repayment, the amount of the annuity is constant during the entire period.

$\overline{A}$ constant annuity, $A_1 = A_2 = ... = A_n = \overline{A}$

The interest portion decreases from period to period, while the repayment portion increases by a corresponding amount. The payment of annuities is equivalent to an annuity in arrears. C_0 is the present value of all annuities.



Annuity $$\overline{A} = C_0 \cdot \underbrace{\frac{q^n \cdot (q-1)}{q^n - 1}}_{\text{annuity factor in arrears}} \quad \text{or} \quad \overline{A} = T_1 \cdot q^n$$

Initial Debt $$C_0 = A \cdot \underbrace{\frac{q^n - 1}{q^n \cdot (q-1)}}_{\text{present value of annuity factor in arrears}} \quad \text{or} \quad C_0 = T_1 \cdot \underbrace{\frac{q^n - 1}{i}}_{\text{amount of annuity factor in arrears}}$$

Residual Debt $$C_k = C_0 \cdot q^k - A \cdot \frac{q^k - 1}{q - 1} \quad \text{or} \quad C_k = C_0 - T_1 \cdot \tfrac{q^k - 1}{q-1}$$

Repayment Instalment $T_k = C_0 \cdot \frac{i \cdot q^{k-1}}{q^n - 1}$ or $T_k = T_1 \cdot (1+i)^{k-1}$ or

$$T_1 = C_0 \cdot \frac{i}{(1+i)^n - 1} \quad \text{or} \quad T_1 = \overline{A} \cdot q^{-n}$$

Interest Amount $Z_k = \overline{A} - T_1 \cdot q^{k-1}$ or $Z_k = C_{k-1} \cdot i$ or

$$Z_k = \overline{A} \cdot (1 - q^{k-1}) + C_0 \cdot i \cdot q^{k-1}$$

Interest Factor $q^n = \frac{\overline{A}}{\overline{A} - C_0 \cdot i} = \frac{\overline{A}}{T_1}$

Repayment Period $n = \frac{\log\left(\frac{\overline{A}}{T}\right)}{\log(q)}$ or $n = \frac{\ln\left(\frac{\overline{A}}{T}\right)}{\ln(q)}$

Example 1: Mrs. Penny has taken out a loan of \$200,000. She commits herself to pay interest at the end of each year at a rate of $3.5\,\%$ p.a. After 6 years she wants the loan to be completely repaid. To keep the annual paid amount constant, Mrs. Penny chooses the annuity repayment.

$$\overline{A} = \$200,000 \cdot \frac{1.035^6 \cdot 0.035}{1.035^6 - 1} = \$37,533.64$$

$$Z_1 = \$200,000 \cdot 0.035 = \$7,000$$

$$T_1 = \$37,533.64 - \$7,000 = \$30,533.64$$

k	Residual Amount at the Beginning of the Year	Interest Amount	Repayment Instalment	Annuity	Residual Amount at the End of the Year
1	200,000.00	7,000.00	30,533.64	37,533.64	169,466.36
2	169,466.36	5,931.32	31,602.32	37,533.64	137,864.04
3	137,864.04	4,825.24	32,708.40	37,533.64	105,155.64
4	105,155.64	3,680.45	33,853.19	37,533.64	71,302.45
5	71,302.45	2,495.59	35,038.05	37,533.64	36,264.40
6	36,264.40	1,269.25	36,264.39	37,533.64	0.01

Note: The difference of $0.01 is due to rounding errors.

Example 2: Your bank grant you a loan of $30,000.00 and agrees with you that the loan is to be repaid in the form of an annuity. The initial repayment rate is 20 % and the annual interest rate is 7 %.

a) Calculate the exact term of the loan in years, including two decimal places.

b) Create an amortization schedule in form of a redemption plan.

a) $\overline{A} = T_1 \cdot q^n \quad \Leftrightarrow \quad T_1 = \dfrac{\overline{A}}{q^n}$

$$T_1 = \frac{6,000 + 2,100}{(1.07)^n}$$

with $6,000 = 30,000 \cdot 0.2\,(20\%)$

and $2,100 = 30,000 \cdot 0.07$

$\Rightarrow \overline{A} = 6,000 + 2,100 = \$8,100$

$$n = ?$$

$$(1{,}07)^n = \frac{8,100}{T_1} = \frac{8,100}{6,000}$$

$$ln(1.07)^n = ln\left(\frac{8,100}{6,000}\right)$$

$$n \cdot ln(1.07) = ln\left(\frac{8,100}{6,000}\right)$$

$$n = \frac{ln(8,100/6,000)}{ln(1.07)} \approx 4.436 \text{ years}$$

b)

k	Account Balance at the Beginning of the Year	Interest Amount	Repayment Instalment	Annuity	Account Balance at the End of the Year
1	30,000	2,100	6,000	8,100	24,000
2	24,000	1,680	6,420	8,100	17,580
3	17,580	1,230.60	6,869.40	8,100	10,710.60
4	10,710.60	749.74	7,350.26	8,100	3,360.34
5	3,360.34	235.22	3,360.34	3,595.56	0

7.5.3 Repayment by Instalments

In the case of repayment by instalments, a debt C_0 is repaid by repayment instalments $\overline{T}$ that remain constant each year.

$\overline{T}$ constant repayment instalment, $T_1 = T_2 = ... = T_n = \overline{T}$

The interest amount to be paid decreases from period to period.

Repayment Instalment

$$\overline{T} = \frac{C_0}{n}$$

Initial Debt

$$C_0 = n \cdot \overline{T}$$

Residual Debt

$$C_k = C_0 - k \cdot \overline{T} \quad \text{or}$$

$$C_k = C_0 \cdot \left(1 - \frac{k}{n}\right)$$

Interest Amount

$$Z_k = i \cdot C_{k-1} \quad \text{or}$$

$$Z_k = i \cdot \left[C_0 - (k-1) \cdot \overline{T}\right] \quad \text{or}$$

$$Z_k = i \cdot C_0 \cdot \left(1 - \frac{k-1}{n}\right)$$

Annuity

$$A_k = \underbrace{i \cdot C_0 \cdot \left(1 - \frac{k-1}{n}\right)}_{Z_k} + \underbrace{\frac{C_0}{n}}_{T_k}$$

$$A_k = \frac{C_0}{n} \cdot \left[1 + (n-k+1) \cdot i\right]$$

Example: Mrs. Penny has a loan of \$150,000 at an interest rate of $2.25\,\%$ p.a. payable in arrears. She agrees to repay the loan in instalments over 5 years.

$$\overline{T} = \frac{\$150,000}{5} = \$30,000$$

$$Z_1 = \$150,000 \cdot 0.0225 = \$3,375$$

k	Residual Amount at the Beginning of the Year	Interest Amount	Repayment Instalment	Annuity	Residual Amount at the End of the Year
1	150,000.00	3,375.00	30,000.00	33,375.00	120,000.00
2	120,000.00	2,700.00	30,000.00	32,700.00	90,000.00
3	90,000.00	2,025.00	30,000.00	32,025.00	60,000.00
4	60,000.00	1,350.00	30,000.00	31,350.00	30,000.00
5	30,000.00	675.00	30,000.00	30,675.00	0.00

7.5.4 Repayment with Premium

If the repayment amount of a loan exceeds the nominal amount, it is referred to as a premium or agio a. The premium refers to the respective repayment period and is expressed as a fixed percentage α. It is also to be repaid. However, no interest may be paid on the premium, as the creditor only receives interest on the nominal amount of the loan.

a	premium or agio
α	premium percentage

The opposite of an agio is a disagio (see chapter 7.5.5).

7.5.4.1 Annuity Repayment with Premium

(a) Annuity repayment with non-included premium

The premium or agio is paid in addition to the annuity. The annuity

(= sum of repayment instalment and interest amount) remains constant over the entire period. The premium or agio increases from year to year, as it is calculated as a percentage of the repayment instalment, which also increases. As a result, the annuity including premium (= sum of repayment instalment, interest amount and premium) increases.

A_α annuity including premium

Example: Mrs. Penny owes the bank a loan of \$100,000, which is to be paid off in 5 years at an interest rate of $1.5\,\%$ p.a. through annuity repayment. The bank charges Mrs. Penny with a premium in form of an agio of $5\,\%$.

$$\overline{A} = C_0 \cdot \frac{q^n \cdot i}{q^n - 1}$$

$$= \$100,000 \cdot \frac{1.015^5 \cdot 0.015}{1.015^5 - 1} = \$20,908.93$$

$$T_1 = C_0 \cdot \frac{q^{k-1} \cdot i}{q^n - 1}$$

$$= \$100,000 \cdot \frac{1.015^0 \cdot 0.015}{1.015^5 - 1} = \$19,408.93$$

$$Z_1 = C_{k-1} \cdot i = \$100,000 \cdot 0.015 = \$1,500$$

alternatively: $Z_1 = \overline{A} - T_1$

$$= \$20,908.93 - \$19,408.93 = \$1,500$$

$$C_1 = C_0 - T_1 \cdot \frac{q^k - 1}{q - 1}$$

$$= \$100,000 - \$19,408.93 \cdot \frac{1.015^1 - 1}{1.015 - 1} = \$80,591.07$$

alternatively: $C_1 = C_0 \cdot q^k - \overline{A} \cdot \dfrac{q^k - 1}{q - 1}$

$$= \$100,000 \cdot 1.015^1 - \$20,908.93 \cdot \frac{1.015^1 - 1}{1.015 - 1}$$

$$= \$80,591.07$$

$$a_1 = T_1 \cdot \alpha = \$19,408.93 \cdot 0.05 = \$970.45$$

$$A_\alpha = \overline{A} + a_1 = \$20,908.93 + \$970.45 = \$21,879.38$$

k	Residual Amount at the Beginning of the Year	Interest Amount	Repayment Instalment	Annuity	Premium	Annuity including Premium
1	100,000.00	1,500.00	19,408.93	20,908.93	970.45	21,879.38
2	80,591.07	1,208.87	19,700.06	20,908.93	985.00	21,893.93
3	60,891.01	913.37	19,995.56	20,908.93	999.78	21,908.71
4	40,895.45	613.43	20,295.50	20,908.93	1,014.78	21,923.71
5	20,599.95	309.00	20,599.93	20,908.93	1,029.99	21,938.92

Note: Differences are due to rounding errors.

(b) Annuity repayment with included premium

The premium or agio is already included in the annuity. This means that the annual amount to be paid remains constant throughout the entire period. One speaks of an annuity with included premium.

$\overline{A}_\alpha$	annuity with premium included
C_α	replacement capital ($=$ redemption value, notional debt)
i_α	Replacement interest rate ($=$ notional interest rate), which, when applied to C_α, results in the same amount of interest as the interest rate i charged on the initial capital C_0.
q_α^n	notional accumulation factor $q_\alpha^n = (1+i_\alpha)^n$

Annuity

$$\overline{A}_\alpha = \frac{i \cdot (1+i_\alpha)^n}{(1+i_\alpha)^n - 1} \cdot C_0 \text{ or}$$

$$\overline{A}_\alpha = (1+i_\alpha)^n \cdot C_\alpha \cdot \frac{(1+i_\alpha)-1}{(1+i_\alpha)^n - 1}$$

Notional Interest Rate

$$i_\alpha = \frac{i}{1+\alpha}$$

Redemption Value

$$C_\alpha = C_0 \cdot (1+\alpha) = \frac{C_0 \cdot i}{i_\alpha}$$

Repayment Instalment with Premium

$$T_k = A_\alpha - i \cdot C_k$$

Repayment Instalment without Premium

$$T_k = \frac{A_\alpha - i \cdot C_k}{1+\alpha}$$

Example: Mrs. Penny owes the bank a loan of \$100,000, which is to be paid off in 5 years at an interest rate of $1.5\,\%$ p.a. through annuity repayment. The premium or agio of $5\,\%$ is to be included in the repayment instalment, so that the total annual payment remains constant during the period.

$$i_\alpha = \frac{0.015}{1+0.05} = 0.01428571429$$

$$\overline{A}_\alpha = \frac{0.015\cdot(1+0.01428571429)^5}{(1+0.01428571429)^5-1}\cdot \$100{,}000$$

$$= \$21{,}908.51$$

$$T_1 = \frac{\$21{,}908.51 - 0.015\cdot\$100{,}000}{1+0.05} = \$19{,}436.68$$

$$Z_1 = \$100{,}000\cdot 0.015 = \$1{,}500.00$$

$$a_1 = \$21{,}908.51 - \$19{,}436.68 - \$1{,}500.00 = \$971.83$$

k	Residual Amount at the Beginning of the Year	Interest Amount	Repayment Instalment without Premium	Premium	Repayment Instalment with Premium	Annuity including Premium
1	100,000	1,500.00	19,436.68	971.83	20,408.51	21,908.51
2	80,563.32	1,208.45	19,714.34	985.72	20,700.06	21,908.51
3	60,848.98	912.73	19,995.98	999.80	20,995.78	21,908.51
4	40,853.00	612.80	20,281.63	1,014.08	21,295.71	21,908.51
5	20,571.37	308.57	20,571.37	1,028.57	21,599.94	21,908.51

7.5.4.2 Repayment of an Instalment Debt with Premium

In the case of repayment by instalments, the repayment amount remains constant over the period. The annuity increases by the premium or agio.

Premium

$$a = \frac{C_0}{n} \cdot \alpha = \overline{T}_k \cdot \alpha$$

Repayment Amount including Premium

$$T_\alpha = (1+\alpha) \cdot \overline{T}_k$$

Annuity including Premium

$$A_k = C_0 \cdot \left[\frac{1}{n} + \left(1 - \frac{k-1}{n}\right) \cdot i + \frac{\alpha}{n}\right] \quad \text{or}$$

$$A_k = \underbrace{C_0 \cdot i \cdot \left(1 - \frac{k-1}{n}\right)}_{Z} + \underbrace{\frac{C_0}{n}}_{T} + \underbrace{\frac{C_0}{n} \cdot \alpha}_{a}$$

$$A_k = Z_k + \overline{T}_k + (1+\alpha)$$

$$A_k = Z_k + \overline{T}_k + a$$

<u>Example:</u> Mrs. Penny owes the bank a loan of \$100,000, which is to be paid off in 5 years by means of repayment in instalments at an interest rate of $1.5\,\%$ p.a. The bank charges Mrs. Penny a premium in form of an agio of $5\,\%$.

$$a = \$20{,}000 \cdot 0.05 = \$1{,}000$$

k	Residual Amount at the Beginning of the Year	Interest Amount	Repayment Instalment	Premium	Annuity including Premium
1	100,000.00	1,500.00	20,000.00	1,000.00	22,500.00
2	80,000.00	1,200.00	20,000.00	1,000.00	22,200.00
3	60,000.00	900.00	20,000.00	1,000.00	21,900.00
4	40,000.00	600.00	20,000.00	1,000.00	21,600.00
5	20,000.00	300.00	20,000.00	1,000.00	21,300.00

7.5.5 Repayment with Discount (Disagio)

A discount (disagio or debt discount) corresponds to a deduction from the face value, which may be agreed in the case of a loan or the issue of a security.

b	discount or disagio
β	discount rate in percent

The opposite of a disagio is the agio (see chapter 7.5.4).

Annuity Repayment with Discount

The amount paid out is reduced by the amount b if a disagio is agreed on. However, the discount, b, is fully included in the the full amount of the repayment amount. The disagio can be interpreted quasi as interest paid in advance. A disagio reduces the nominal interest rate for the periodic instalment during the entire period of the loan.

A disagio can take the form of the amount paid or the issue price. (e.g., when securities are issued) in monetary units (e.g. $) or as a percentage of the loan amount: 5 %. disagio, 95 % amount paid or issue price.

For a loan with a disagio, only the loan amount reduced by the disagio is disbursed, not the full loan amount. For example, with a loan amount of $100,000 and a disagio of 5 %, the amount paid is $95,000.

However, the repayment sum is equal to the amount of the total loan sum (100 % =$100,000). Accordingly, this liability must be reported in the balance sheet in the full amount of the total loan sum, in this case over $100,000.

Legally, the accounting for a disagio differs internationally. Often, as for example in Germany,[16] there is an option to capitalize the disagio for companies required to prepare financial statements,

a) to record the disagio immediately, i.e. at the time the loan is granted, in the full amount as an expense in the income statement, or

b) to record prepaid expenses[17] within the balance sheet and to distribute it by scheduled depreciation over the entire period of the loan.

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[16] In Germany, the accounting of a disagio is regulated in § 250 para. 3 HGB.

[17] In Germany pursuant to sec. § 266 para. 2 C. HGB (“prepaid expenses“).

7.5.5.1 Annuity Repayment with Discount when Immediately Booked as Interest Expense

If the disagio is booked as interest expense immediately, the amount of the discount, b, is attributed to that one period only.

Example:

A company takes out a loan for \$100,000 with a period of five years. A disagio of $\beta = 5\ \%$ is agreed with the lending bank. The loan is to be repaid in five years at an interest rate of 1.5 % p.a. on the amount paid by annuity repayment.

Since the company needs the \$100,000 in full, the lender increases the principal amount to \$105,263.16 due to the disagio.[18]

$$\overline{A} = K_0 \cdot \frac{q^n \cdot i}{q^n - 1} =$$

$$= \$100{,}000 \cdot \frac{1.015^5 \cdot 0.015}{1.015^5 - 1} = \$20{,}908.93$$

$$T_1 = K_0 \cdot \frac{q^{k-1} \cdot i}{q^n - 1} =$$

$$= \$100{,}000 \cdot \frac{1.015^0 \cdot 0.015}{1.015^5 - 1} = \$19{,}408.93$$

$$Z_1 = K_{k-1} \cdot i = \$100{,}000 \cdot 0.015 = \$1{,}500$$

alternatively: $Z_1 = \overline{A} - T_1 =$

---

[18] $\frac{\$100{,}000}{0.95} = \$105{,}263.16$ or $\$105{,}263.16 \cdot 0.95 = \$100{,}000$

$$= \$20{,}908.93 - \$19{,}408.93 = \$1{,}500$$

$$K_1 = K_0 - T_1 \cdot \frac{q^k - 1}{q - 1} =$$

$$= \$100{,}000 - \$19{,}408.93 \cdot \frac{1.015^1 - 1}{1.015 - 1} = \$80{,}591.07$$

alternatively: $K_1 = K_0 \cdot q^k - \overline{A} \cdot \dfrac{q^k - 1}{q - 1} =$

$$= \quad \$100{,}000 \quad \cdot \quad 1.015^1 \quad - \quad \$20{,}908.93 \quad \cdot$$

$$\cdot \quad \frac{1.015^1 - 1}{1.015 - 1} = \$80{,}591.07$$

$$\beta = K_0 \cdot \beta = \$100{,}000 \cdot 0.05 = \$5{,}000.00$$

k	**Residual Amount at the Beginning of the Year**	**Interest Amount**	**Repay-ment Instal-ment**	**Annuity**	**Discount**	**Expenses Including Discount**
1	100,000.00 (amount paid)	1,500.00	19,408.93	20,908.93	5,263.16	26,172.09
2	80,591.07	1,208.87	19,700.06	20,908.93		20,908.93
3	60,891.01	913.37	19,995.56	20,908.93		20,908.93
4	40,895.45	613.43	20,295.50	20,908.93		20,908.93
5	20,599.95	309.00	20,599.93	20,908.93		20,908.93

Differences are due to rounding errors.

7.5.5.2 Annuity Repayment with Discount when a Disagio is Included in Prepaid Expenses

Are the legal requirements for such an action met[19] the difference between the loan amount and the amount paid (disagio) is to be amortized by annual scheduled depreciation, which can be spread over the entire annual period of the liability.[20]

For the example above the following is valid:

k	**Residual Amount at the Beginning of the Year**	**Interest Amount**	**Repayment Instalment**	**Annuity**	**Discount (linearly distributed)[21]**	**Expenses Including Discount**
1	100,000.00	1,500.00	19,408.93	20,908.93	1,052.63	21,961.56
2	80,591.07	1,208.87	19,700.06	20,908.93	1,052.63	21,961.56
3	60,891.01	913.37	19,995.56	20,908.93	1,052.63	21,961.56
4	40,895.45	613.43	20,295.50	20,908.93	1,052.63	21,961.56
5	20,599.95	309.00	20,599.93	20,908.93	1,052.63	21,961.56

Differences are due to rounding errors.

7.5.5.3 Instalment Repayment with Discount when Immediately Booked as Interest Expense

In the case of instalment repayment, the repayment amount remains constant over the entire period.

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[19] In Germany pursuant to § 250 (3) HGB.

[20] In Germany pursuant to § 250 (3) sentence 2 HGB.

[21] $\$5,263.16 / 5 = \$1,052.63$

Example:

A company takes out a loan at \$100,000 with a period of five years. A discount of $\beta = 5$ % is agreed with the lending bank. The loan is to be repaid in five years at an interest rate of 1.5 % on the amount paid p.a. through instalment repayments.

Since the company needs the \$100,000 in full, the lender increases the nominal amount to \$105,263.16 due to the disagio.[22]

k	**Residual Amount at the Beginning of the Year**	**Interest Amount**	**Repayment Instalment**	**Discount**	**Expenses Including Discount**
1	100,000.00	1,500.00	20,000.00	5,263.16	26,500.00
2	80,000.00	1,200.00	20,000.00		21,200.00
3	60,000.00	900.00	20,000.00		20,900.00
4	40,000.00	600.00	20,000.00		20,600.00
5	20,000.00	300.00	20,000.00		20,300.00

Differences are due to rounding errors.

7.5.5.4 Instalment Repayment with Discount when a Disagio is Included in Prepaid Expenses

Are the legal requirements for such an action met[23] the difference between the loan amount and the amount paid (disagio) is to be amortized by annual scheduled depreciation, which can be spread over the entire annual period of the liability. [24].

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[22] $\frac{\$100,000}{0.95} = \$105,263.16$ or $\$105,263.16 \cdot 0.95 = \$100,000$

[23] In Germany pursuant to § 250 (3) HGB,

[24] In Germany pursuant to § 250 (3) sentence 2 HGB

For the example above the following is valid:

k	Residual Amount at the Beginning of the Year	Interest Amount	Repayment Instalment	Discount (linearly distributed)	Expenses Including Discount
1	100,000.00	1,500.00	20,000.00	1,052.63	22,552.63
2	80,000.00	1,200.00	20,000.00	1,052.63	22,252.63
3	60,000.00	900.00	20,000.00	1,052.63	21,952.63
4	40,000.00	600.00	20,000.00	1,052.63	21,652.63
5	20,000.00	300.00	20,000.00	1,052.63	21,352.63

7.5.6 Grace Periods

If the credit period n_p exceeds the repayment period n_r, i.e. $n_p > n_r$, this is referred to as a grace period. Within this grace period, only interest is payable on the debt. Repayment is suspended, thus the burden on the borrower for this period is reduced.

The following applies: $T_k = 0$ if $k \leq n_p - n_r$

n_p	credit period in years
n_r	repayment period in years

Example:

(1) Grace Periods for Annuity Repayment

Camilla takes out a loan of \$50,000. Since she is still in vocational training, she would like to postpone repayment for the first three years. After that she can repay the loan within 5 years. The interest rate is $4\,\%$ p.a.

$n_p = 8; \;\; n_r = 5$

The repayment period n_r is key to calculating the annuity.

$$\overline{A} = \$50{,}000 \cdot \frac{1.04^5 \cdot 0.04}{1.04^5 - 1} = \$11{,}231.36$$

k	**Residual Amount at the Beginning of the Year**	**Interest Amount**	**Repayment Instalment**	**Annuity**
1	50,000.00	2,000.00	0.00	2,000.00
2	50,000.00	2,000.00	0.00	2,000.00
3	50,000.00	2,000.00	0.00	2,000.00
4	50,000.00	2,000.00	9,231.36	11,231.36
5	40,768.64	1,630.75	9,600.61	11,231.36
6	31,168.03	1,246.72	9,984.64	11,231.36
7	21,183.39	847.34	10,384.02	11,231.36
8	10,799.37	431.97	10,799.37	11,231.36

(2) Grace Periods for Repayment by Instalments

Camilla would like to know what the annual burden would be if she were to agree to an instalment repayment instead of an annuity repayment.

$$\overline{T}_k = \frac{\$50{,}000}{5} = \$10{,}000$$

k	Residual Amount at the Beginning of the Year	Interest Amount	Repayment Instalment	Annuity
1	50,000.00	2,000.00	0.00	2,000.00
2	50,000.00	2,000.00	0.00	2,000.00
3	50,000.00	2,000.00	0.00	2,000.00
4	50,000.00	2,000.00	10,000.00	12,000.00
5	40,000.00	1,600.00	10,000.00	11,600.00
6	30,000.00	1,200.00	10,000.00	11,200.00
7	20,000.00	800.00	10,000.00	10,800.00
8	10,000.00	400.00	10,000.00	10,400.00

7.5.7 Rounded Annuities

In some cases, rounded amounts are desired as annuity, e.g. for percentage annuity and for the repayment of bonds. The calculation therefore deviates from the strict ideal of constant annuities over the entire period.

7.5.7.1 Percentage Annuity

With percentage annuity, the repayment rate i_T in the first year is given as a percentage of the debt C_0. The annuity is calcualted together with the interest rate i. The interest saved in the subsequent years is used for repayment.

$T = i_T \cdot C_0$ The repayment amount of the 1st year.

This results in crooked periods. Therefore the period is broken down into two components n_1 and n_2. A compensation payment is due because the period does not occur in full years. If the compensation payment is paid at the end of the period, it is called a final payment. If it is paid at the beginning of the period, it is called an advance payment.

i_T	repayment rate of the 1st year
n_1	$n_1 = int(n)$
n_2	$n_2 = n - n_1$

Annuity $A = C_0 = (i + i_T)$

Period $n = \dfrac{\ln\left(\dfrac{i+i_T}{i_T}\right)}{\ln(q)}$ with $n \notin Z$

Final Payment $A_{n_1+1} = \left(C_0 \cdot q^{n_1} - A \cdot \dfrac{q^{n_1}-1}{i}\right) \cdot q$

Advance Payment $A_1 = C_0 \cdot q - A \cdot \dfrac{q^{n_1}-1}{i \cdot q^{n_1}}$

Example:

Camilla takes out a loan of \$120,000. The loan is to be repaid in the form of a percentage annuity repayment. The initial repayment rate is $7\,\%$, the annual interest rate $3\,\%$ p.a.

$C_0 = \$120{,}000;\ \ i = 0.03\,(3\,\%);\ \ i_T = 0.07\,(7\,\%)$

$A = \$120{,}000 \cdot (0.03 + 0.07) = \$12{,}000$

$$n = \frac{\ln\left(\dfrac{0.03+0.07}{0.07}\right)}{\ln(1.03)} = 12.06 \text{ years}$$

(1) Final Payment

Camilla makes the compensation payment at the end of the period.

$$A_{13} = \left(\$120,000 \cdot 1.03^{12} - \$12,000 \cdot \frac{1.03^{12} - 1}{0.03}\right) \cdot 1.03$$

$$= \$810.56$$

k	Residual Amount at the Beginning of the Year	Interest Amount	Repayment Instalment	Annuity
1	120,000.00	3,600.00	8,400.00	12,000.00
2	111,600.00	3,348.00	8,652.00	12,000.00
3	102,948.00	3,088.44	8,911.56	12,000.00
4	94,036.44	2,821.09	9,178.91	12,000.00
5	84,857.53	2,545.73	9,454.27	12,000.00
6	75,403.26	2,262.10	9,737.90	12,000.00
7	65,665.36	1,969.96	10,030.04	12,000.00
8	55,635.32	1,669.06	10,330.94	12,000.00
9	45,304.38	1,359.13	10,640.87	12,000.00
10	34,663.51	1,039.91	10,960.09	12,000.00
11	23,703.42	711.10	11,288.90	12,000.00
12	12,414.52	372.44	11,627.56	12,000.00
13	786.96	23.61	786.96	810.56

(2) Advance Payment

Camilla makes the compensation payment at the beginning of the period.

$$\begin{aligned} A_1 &= \$120,000 \cdot 1.03 - \$12,000 \cdot \frac{1.03^{12}-1}{0.03 \cdot 1.03^{12}} \\ &= \$4,151.95 \end{aligned}$$

k	Residual Amount at the Beginning of the Year	Interest Amount	Repayment Instalment	Annuity
1	120,000.00	3,600.00	551.95	4,151.95
2	119,448.05	3,583.44	8,416.56	12,000.00
3	111,031.49	3,330.94	8,669.06	12,000.00
4	102,362.43	3,070.87	8,929.13	12,000.00
5	93,433.30	2,803.00	9,197.00	12,000.00
6	84,236.30	2,527.09	9,472.91	12,000.00
7	74,763.39	2,242.90	9,757.10	12,000.00
8	65,006.29	1,950.19	10,049.81	12,000.00
9	54,956.48	1,648.69	10,351.31	12,000.00
10	44,605.17	1,338.16	10,661.84	12,000.00
11	33,943.33	1,018.30	10,981.70	12,000.00
12	22,961.63	688.85	11,311.15	12,000.00
13	11,650.48	349.51	11,650.48	12,000.00

7.5.7.2 Repayment of Bonds

When companies need to fund investments, it is often the case that the required loan amount cannot be raised by a single creditor. The company then issues bonds so that a large number of creditors can participate.

The bonds issued are divided into rounded partial amounts w (e.g. \$100, \$500, \$1,000, \$5,000 and \$10,000).

The calculated repayment instalments T_k are broken down into round partial amounts (final, rounded repayment instalment T_k^* and repayment arrears R_k). Only repayment instalments that can be divided by the partial amounts without a remainder are allowed. Partial repayment of an integer unit of measure is not possible.

w	value per bond
pi_k	number of bonds repaid
T_k^*	final repayment instalment
R_k	repayment arrears
a_k	number of bonds to be repaid

Annuity

$$A = C_0 \cdot \frac{i \cdot q^n}{q^n - 1}$$

Interest Amount

$$Z_k = i \cdot C_{k-1}$$

Preliminary Repayment Instalment

$$T_k = A - Z_k \qquad \text{if} \quad k = 1$$

$$T_k = A - Z_k + (1+i) \cdot R_{k-1} \qquad \text{if} \quad k > 1$$

Number of Bonds to be Repaid

$$a_k = int\left(\frac{T_k}{w}\right)$$

Final Repayment Instalment

$$T_k^* = w \cdot a_k$$

Repayment Arrears $R_k = T_k - T_k^*$

Residual Debt $C_k = C_{k-1} - T_k^*$

Examples:

(1) Repayment of Bonds with Equal Denomination

In bonds with equal denominations, the individual securities have the same nominal value.

$C_0 = \$5,000,000$; $i = 0.06\,(6\,\%)$; $n = 5$; $5,000$ bonds;
$w = \$1,000$ per bond

$$A = \$5,000,000 \cdot \frac{0.06 \cdot 1.06^5}{1.06^5 - 1} = \$1,186,982.00$$

$$Z_1 = 0.06 \cdot \$5,000,000 = \$300,000$$

$$T_1 = \$1,186,982.00 - \$300,000 = \$886,982.00$$

$$a_1 = int\left(\frac{\$886,982}{\$1,000}\right) = 886$$

$$T_k^* = 886 \cdot \$1,000 = \$886,000$$

$$C_1 = \$5,000,000 - \$886,000 = \$4,114,000$$

$$R_1 = \$886,982 - \$886,000 = \$982.00$$

$$T_2 = \underbrace{\$1,186,982.00}_{A} - \underbrace{\$246,840}_{Z_2} + \underbrace{(1+0.06)\cdot\$982.00}_{(1+i)\cdot R_1} = \$941,182.92$$

k	C_{k-1}	Z_k	T_k	a_k	T_k^*	R_l	A_k
1	5,000,000.00	300,000.00	886,982.00	886	886,000.00	982.00	1,186,000.00
2	4,114,000.00	246,840.00	941,183.00	941	941,000.00	182.92	1,187,840.00
3	3,173,000.00	190,380.00	996,796.00	996	996,000.00	795.90	1,186,380.00
4	2,177,000.00	130,620.00	1,057,206.00	1,057	1,057,000.00	205.66	1,187,620.00
5	1,120,000.00	67,200.00	1,120,000.00	1,120	1,120,000.00	0.00	1,187,200.00

(2) Repayment of Bonds with Unequal Denomination

In bonds with unequal denominations, the individual securities have different nominal values.

$C_0 = \$5,000,000;\;\; i = 0.06\,(6\,\%);\;\; n = 5$

Denomination: a) 250 bonds at $10,000 each

b) 500 bonds at $5,000 each

Repayment Schedule with Denomination for Partial Bond a):

k	C_{k-1}	Z_k	T_k	a_k	T_k^*	R_l	A_k
1	2,500,000.00	150,000.00	443,491.00	44	440,000.00	3,491.00	590,000.00
2	2,060,000.00	123,600.00	473,591.46	47	470,000.00	3,591.46	593,600.00
3	1,590,000.00	95,400.00	501,897.95	50	500,000.00	1,897.95	595,400.00
4	1,090,000.00	65,400.00	530,102.83	53	530,000.00	102.83	595,400.00
5	560,000.00	33,600.00	560,000.00	56	560,000.00	0	593,600.00
				250	2,500,000		

Repayment Schedule with Denomination for Partial Bond b):

k	C_{k-1}	Z_k	T_k	a_k	T_k^*	R_l	A_k
1	2,500,000.00	150,000.00	443,491.00	88	440,000.00	3,491.00	590,000.00
2	2,060,000.00	123,600.00	473,591.46	94	470,000.00	3,591.46	593,600.00
3	1,590,000.00	95,400.00	501,897.95	100	500,000.00	1,897.95	595,400.00
4	1,090,000.00	65,400.00	530,102.83	106	530,000.00	102.83	595,400.00
5	560,000.00	33,600.00	560,000.00	112	560,000.00	0	593,600.00
				500	2,500,000		

Total Repayment Schedule for Partial Bounds a) and b):

$$A_{a/b} = \$2,500,000 \cdot \frac{1.06^5 \cdot 0.06}{1.06^5 - 1} = \$593,491.00$$

$$T_{1a/b} = \$593,491 - \$150,000 = \$443,491.00$$

$$T_{2a/b} = \$593,491 - \$123,600 + 1.06 \cdot \$3,491 = \$473,591.46$$

$$\vdots$$

$$T_{5a/b} = \$593,491 - \$33,600 + 1.06 \cdot \$102.83 = \$560,000.00$$

$$a_{1a} = int\left(\frac{\$443,491}{\$10,000}\right) = 44 \qquad a_{1b} = int\left(\frac{\$443,491}{\$5,000}\right) = 88$$

$$\vdots$$

$$a_{5a} = int\left(\frac{\$560,000}{\$10,000}\right) = 56 \qquad a_{5b} = int\left(\frac{\$560,000}{\$5,000}\right) = 112$$

k	C_{k-1}	Z_k	T_k^*	A_k	a_k	
					a)	b)
1	500,000.00	300,000.00	886,000.00	1,186,000.00	44	88
2	4,114,000.00	246,840.00	941,000.00	1,187,840.00	47	94
3	3,173,000.00	190,380.00	996,000.00	1,186,380.00	50	100
4	2,177,000.00	130,620.00	1,057,000.00	1,187,620.00	53	106
5	1,120,000.00	67,200.00	1,120,000.00	1,187,200.00	56	112
		935,040.00	5,000,000.00	5,935,040.00	250	500

7.5.8 Repayment During the Year

If there are several repayment and/or interest periods within one year, this is known as repayment during the year.

m_r	number of repayment periods per year
m_z	number of interest periods per year
N	total period of a loan/credit measured in repayment periods
N_1	$N_1 = n_1 \cdot m_r$
N_2	$N_2 = n_2 \cdot m_r$

7.5.8.1 Annuity Repayment During the Year

In the case of annuity repayments during the year, the burden of the repayment instalment and interest amount is constant for each repayment period during the year.

The following applies: $A_1 = A_2 = A_3 = ... = A_n = \overline{A}$

(a) Equal number of interest and repayment periods $(m_z = m_r)$

Relative Interest Rate $j = \dfrac{i}{m_z}$

Annuity $\overline{A} = C_0 \cdot \dfrac{j \cdot (1+j)^N}{(1+j)^N - 1}$

Period of a Loan $N = n \cdot m_r$ (measured in repayment periods)

Example: Camilla takes out a loan of \$12,000 at an interest rate of $5\,\%$ p.a. Interest and repayment payments are made quarterly. The loan is expected to be repaid after three years.

$C_0 = \$12{,}000;\ \ i = 0.05\,(5\,\%);\ \ n = 3;\ \ m_z = m_r = 4$

$N = 3 \cdot 4 = 12$ quarters (repayment periods)

$$j = \frac{0.05}{4} = 0.0125$$

$$\overline{A} = \$12{,}000 \cdot \frac{0.0125 \cdot (1 + 0.0125)^{12}}{(1 + 0.0125)^{12} - 1} = \$1{,}083.10$$

Year	Quarter	Residual Amount at the Beginning of the Year	Interest Amount	Repayment Instalment	Annuity
1	1	12,000.00	150.00	933.10	1,083.10
	2	11,066.90	138.34	944.76	1,083.10
	3	10,122.14	126.53	956.57	1,083.10
	4	9,165.57	114.57	968.53	1,083.10
2	1	8,197.04	102.46	980.64	1,083.10
	2	7,216.40	90.21	992.89	1,083.10
	3	6,223.51	77.79	1,005.31	1,083.10
	4	5,218.20	65.23	1,017.87	1,083.10
3	1	4,200.33	52.50	1,030.60	1,083.10
	2	3,169.73	39.62	1,043.48	1,083.10
	3	2,126.25	26.58	1,056.52	1,083.10
	4	1,069.73	13.37	1,069.73	1,083.10

(b) More interest than repayment periods $(m_z > m_r)$

Relative Interest Rate $$j = \left(1 + \frac{i}{m_z}\right)^{\frac{m_z}{m_r}} - 1$$

Annuity $$\overline{A} = C_0 \cdot \frac{j \cdot (1+j)^N}{(1+j)^N - 1}$$

Period of a Loan $N = n \cdot m_r$ (measured in repayment periods)

Example: Camilla takes out a loan of \$12,000 at an interest rate of $5\,\%$ p.a. Interest is paid monthly, while the repayment is made quarterly. The loan should be repaid after three years.

$$C_0 = \$12{,}000;\;\; i = 0.05\,(5\,\%);\;\; n = 3;\;\; m_r = 4;\;\; m_z = 12$$

$N = 3 \cdot 4 = 12$ quarters (repayment periods)

$$j = \left(1 + \frac{0.05}{12}\right)^{\frac{12}{4}} - 1 = 0.0126$$

$$\overline{A} = \$12{,}000 \cdot \frac{0.0126 \cdot (1+0.0126)^{12}}{(1+0.0126)^{12}-1} = \$1{,}083.78$$

Year	Quarter	Residual Amount at the Beginning of the Year	Interest Amount	Repayment Instalment	Annuity
1	1	12,000.00	151.20	932.58	1,083.78
	2	11,067.42	139.45	944.33	1,083.78
	3	10,123.09	127.55	956.23	1,083.78
	4	9,166.86	115.50	968.28	1,083.78
2	1	8,198.58	103.30	980.48	1,083.78
	2	7,218.10	90.95	992.83	1,083.78
	3	6,225.27	78.44	1,005.34	1,083.78
	4	5,219.93	65.77	1,018.01	1,083.78
3	1	4,201.92	52.94	1,030.84	1,083.78
	2	3,171.08	39.96	1,043.82	1,083.78
	3	2,127.26	26.80	1,056.98	1,083.78
	4	1,070.28	13.49	1,070.28	1,083.77

Note: Differences are due to rounding errors.

(c) More repayment than interest periods $(m_r > m_z)$

If there are more repayment periods than interest periods, interest payments no longer accrue at the end of each repayment period, but only if

- an interest period has ended.
- the end of the period has been reached.

Calculated interest at the end of the kth repayment period

$$Z_k^* = \frac{i}{m_r} \cdot C_{k-1}$$

Interest payments at the end of a repayment period

$$Z_k = \left\{ \begin{array}{lll} 0 & \text{if } \dfrac{k}{m_r} \neq int\left(\dfrac{k}{m_r}\right) & \text{and } k < N \\ \displaystyle\sum_{\tau=k-m_r+1}^{k} Z_\tau^* & \text{if } \dfrac{k}{m_r} = int\left(\dfrac{k}{m_r}\right) & \\ \displaystyle\sum_{\tau=n_1 \cdot m_r+1}^{N} Z_\tau^* & \text{if } \dfrac{k}{m_r} \neq int\left(\dfrac{k}{m_r}\right) & \text{and } k = N \end{array} \right\}$$

Annuity

$$\overline{A} = C_0 \cdot \frac{q^{n_1} \cdot (1 + n_2 \cdot i)}{\left(m_r + \frac{i}{2} \cdot (m_r - 1)\right) \cdot \frac{q^{n_1} - 1}{i} \cdot (1 + n_2 \cdot i) + \left(N_2 + \frac{i}{m_r} \cdot \frac{(N_2 - 1) \cdot N_2}{2}\right)}$$

Period components

$$n_1 = int(n) \qquad n_2 = n - n_1$$

$$N_1 = n_1 \cdot m_r \qquad N_2 = n_2 \cdot m_r$$

Example:

Nawid, Camilla's cousin, takes out a loan of \$12,000 at an interest rate of $5\,\%$ p.a. Repayments are made quarterly while interest is paid annually. The loan is scheduled to be repaid after 2.5 years.

$C_0 = \$12,000;\;\; i = 0.05\,(5\,\%);\;\; n_1 = 2;\;\; n_2 = 0.5;\;\; m_r = 4;\;\; m_z = 1$

$N_1 = 2 \cdot 4 = 8$ quarters (repayment periods)

$N_2 = 0.5 \cdot 4 = 2$ quarters (repayment periods)

$N = 2.5 \cdot 4 = 10$ quarters (repayment periods)

$$\overline{A} = \$12,000 \cdot \frac{1.05^2 \cdot (1 + 0.5 \cdot 0.05)}{\left(4 + \frac{0.05}{2} \cdot (4-1)\right) \cdot \frac{1.05^2 - 1}{0.05} \cdot (1 + 0.5 \cdot 0.05) + \left(2 + \frac{0.05}{4} \cdot \frac{(2-1) \cdot 2}{2}\right)}$$

$$= \$12,000 \cdot \frac{1.1300625}{10.57509375} = \$1,282.33$$

$$Z_1^* = \frac{0.05}{4} \cdot \$12,000 = \$150$$

$Z_2 = 0$, since $\frac{2}{4} \neq int\left(\frac{2}{4}\right)$ and $2 < 10$

$$Z_4 = \underbrace{\$150}_{Z_1^*} + \underbrace{\$133.97}_{Z_2^*} + \underbrace{\$117.94}_{Z_3^*} + \underbrace{\$101.91}_{Z_4^*} = \$503.82$$

since $\frac{4}{4} = int\left(\frac{4}{4}\right)$

$$Z_8 = \underbrace{\$92.18}_{Z_5^*} + \underbrace{\$76.15}_{Z_6^*} + \underbrace{\$60.12}_{Z_7^*} + \underbrace{\$44.09}_{Z_8^*} = \$272.54$$

since $\frac{8}{4} = int\left(\frac{8}{4}\right)$

$$Z_{10} = \underbrace{\$31.47}_{Z_9^*} + \underbrace{\$15.44}_{Z_{10}^*} = \$46.91,$$

since $\frac{10}{4} \neq int\left(\frac{10}{4}\right)$ and $k = N \quad (10 = 10)$

Year	Quarter	Residual Amount at the Beginning of the Year	Interest		Repayment Instalment	Annuity
			calcul.	payment		
1	1	12,000.00	150.00	0.00	1,282.33	1,282.33
	2	10,717.67	133.97	0.00	1,282.33	1,282.33
	3	9,435.34	117.94	0.00	1,282.33	1,282.33
	4	8,153.01	101.91	503.82	778.51	1,282.33
2	1	7,374.50	92.18	0.00	1,282.33	1,282.33
	2	6,092.17	76.15	0.00	1,282.33	1,282.33
	3	4,809.84	60.12	0.00	1,282.33	1,282.33
	4	3,527.51	44.09	272.54	1,009.79	1,282.33
3	1	2,517.72	31.47	0.00	1,282.33	1,282.33
	2	1,235.39	15.44	46.91	1,235.39	1,282.30

Note: Differences are due to rounding errors.

7.5.8.2 Repayment by Instalments During the Year

In the case of repayment by instalments during the year, the repayment instalments remain constant for each repayment period during the year.

The following is valid: $T_1 = T_2 = T_3 = ... = T_n = \overline{T}$

Repayment Instalment $\overline{T} = \frac{C_0}{N}$

Period of a Loan $N = n \cdot m_r$ (measured in repayment periods)

(a) Equal number of interest and repayment periods $(m_r = m_z)$

Relative Interest Rate $j = \frac{i}{m_z}$

Example: Camilla takes out a loan of $12,000 at an interest rate of $5\,\%$ p.a. The loan has to be paid off in the form of an instalment repayment. The interest payments and repayments are made quarterly. The loan is scheduled to be repaid after three years.

$C_0 = \$12{,}000;\ \ i = 0.05\,(5\,\%);\ \ n = 3;\ \ m_r = m_z = 4$

$N = 3 \cdot 4 = 12$ quarters (repayment periods)

$$\overline{T} = \frac{\$12{,}000}{12} = \$1{,}000 \qquad j = \frac{0.05}{12} = 0.0125$$

Year	Quarter	Residual Amount at the Beginning of the Year	Interest Amount	Repayment Instalment	Annuity
1	1	12,000.00	150.00	1,000.00	1,150.00
	2	11,000.00	137.50	1,000.00	1,137.50
	3	10,000.00	125.00	1,000.00	1,125.00
	4	9,000.00	112.50	1,000.00	1,112.50
2	1	8,000.00	100.00	1,000.00	1,100.00
	2	7,000.00	87.50	1,000.00	1,087.50
	3	6,000.00	75.00	1,000.00	1,075.00
	4	5,000.00	62.50	1,000.00	1,062.50
3	1	4,000.00	50.00	1,000.00	1,050.00
	2	3,000.00	37.50	1,000.00	1,037.50
	3	2,000.00	25.00	1,000.00	1,025.00
	4	1,000.00	12.50	1,000.00	1,012.50

(b) More interest than repayment periods $(m_z > m_r)$

Relative Interest Rate $j = \left(1 + \frac{i}{m_z}\right)^{\frac{m_z}{m_r}} - 1$

Example: Camilla takes out a loan of \$12,000 at an interest rate of $5\,\%$ p.a. The loan is to be paid off in the form of an instalment repayment. The interest payments are made monthly while the repayments are made quarterly. The loan is to be repaid after three years.

$$C_0 = \$12{,}000;\quad i = 0.05\,(5\,\%);\quad n = 3;\quad m_r = 4;\quad m_z = 12$$

$N = 3 \cdot 4 = 12$ quarters (repayment periods)

$$\overline{T} = \frac{\$12,000}{12} = \$1,000$$

$$j = \left(1 + \frac{0.05}{12}\right)^{\frac{12}{4}} - 1 = 0.01255$$

Year	**Quarter**	**Residual Amount at the Beginning of the Year**	**Interest Amount**	**Repayment Instalment**	**Annuity**
1	1	12,000.00	150.60	1,000.00	1,150.60
	2	11,000.00	138.05	1,000.00	1,138.05
	3	10,000.00	125.50	1,000.00	1,125.50
	4	9,000.00	112.95	1,000.00	1,112.95
2	1	8,000.00	100.40	1,000.00	1,100.40
	2	7,000.00	87.85	1,000.00	1,087.85
	3	6,000.00	75.30	1,000.00	1,075.30
	4	5,000.00	62.75	1,000.00	1,062.75
3	1	4,000.00	50.20	1,000.00	1,050.20
	2	3,000.00	37.65	1,000.00	1,037.65
	3	2,000.00	25.10	1,000.00	1,025.10
	4	1,000.00	12.55	1,000.00	1,012.55

(c) More repayment than interest periods ($m_r > m_z$)

If there are more repayment periods than interest periods, the interest payments no longer accrue at the end of each repayment period. Interest payments only accrue at the end of each repayment period if

- an interest period has ended or
- the end of the period is reached.

Calculated interest at the end of the k^th^ repayment period

$$Z_k^* = \frac{i}{m_r} \cdot C_{k-1}$$

Interest payments at the end of a repayment period

$$Z_k = \left\{ \begin{array}{lll} 0 & \text{if } \dfrac{k}{m_r} \neq int\left(\dfrac{k}{m_r}\right) & \text{and } k < N \\ \sum\limits_{\tau=k-m_r+1}^{k} Z_\tau^* & \text{if } \dfrac{k}{m_r} = int\left(\dfrac{k}{m_r}\right) & \\ \sum\limits_{\tau=n_1 \cdot m_r+1}^{N} Z_\tau^* & \text{if } \dfrac{k}{m_r} \neq int\left(\dfrac{k}{m_r}\right) & \text{and } k = N \end{array} \right\}$$

Example: Camilla takes out a loan of \$12,000 at an interest rate of $5\,\%$ p.a. The loan is to be paid off in the form of an instalment repayment. Repayments are made quarterly while interest payments are made annually. The loan is scheduled to be repaid after 2.5 years.

$$C_0 = \$12{,}000; \;\; i = 0.05\,(5\,\%); \;\; n = 2.5; \;\; m_r = 4; \;\; m_z = 1$$

$N = 2.5 \cdot 4 = 10$ quarters (repayment periods)

$$\overline{T} = \frac{\$12,000}{10} = \$1,200$$

$$Z_1^* = \frac{0.05}{4} \cdot \$12,000 = \$150$$

$$Z_4 = \underbrace{\$150}_{Z_1^*} + \underbrace{\$135}_{Z_2^*} + \underbrace{\$120}_{Z_3^*} + \underbrace{\$105}_{Z_4^*} = \$510$$

since $\frac{4}{4} = int\left(\frac{4}{4}\right)$

$$Z_8 = \underbrace{\$90}_{Z_5^*} + \underbrace{\$75}_{Z_6^*} + \underbrace{\$60}_{Z_7^*} + \underbrace{\$45}_{Z_8^*} = \$270$$

since $\frac{8}{4} = int\left(\frac{8}{4}\right)$

$$Z_{10} = \underbrace{\$30}_{Z_9^*} + \underbrace{\$15}_{Z_{10}^*} = \$45$$

since $\frac{10}{4} \neq int\left(\frac{10}{4}\right)$ and $k = N \quad (10 = 10)$

Year	Quarter	Residual Amount at the Beginning of the Year	Interest		Repayment Instalment	Annuity
			calcul.	payment		
1	1	12,000.00	150.00	0.00	1,200.00	1,200.00
	2	10,800.00	135.00	0.00	1,200.00	1,200.00
	3	9,600.00	120.00	0.00	1,200.00	1,200.00
	4	8,400.00	105.00	510.00	1,200.00	1,710.00
2	1	7,200.00	90.00	0.00	1,200.00	1,200.00
	2	6,000.00	75.00	0.00	1,200.00	1,200.00
	3	4,800.00	60.00	0.00	1,200.00	1,200.00
	4	3,600.00	45.00	270.00	1,200.00	1,470.00
3	1	2,400.00	30.00	0.00	1,200.00	1,200.00
	2	1,200.00	15.00	45.00	1,200.00	1,245.00

7.6 Investment Calculation

In the investment calculation, alternative investment projects are assessed and compared to each other. For this purpose, the information relevant to the investment decision is condensed into an indicator, thus enabling a recommendation to be made for one of the investment alternatives.

A distinction is made between static and dynamic methods:

Static Methods	Dynamic Methods
• Cost comparison calculation • Profit comparison calculation • Amortisation calculation • Return on investment	• Annuity method • Net present value method • Return on investment

7.6.1 Fundamental Terms

Present Value of Capital C_0	The present value of capital corresponds to the value of an investment at the point in time t_0. It is determined by adding the balances of expected incoming and outgoing payments ($=$ series of payments) discounted at the calculatory interest rate i and related to the valuation time t_0.
Amount of Capital C_n	The amount of capital is the profit or loss caused by an investment. It corresponds to the sum of the balances of incoming and outgoing payments ($=$ series of payments) accrued with the calculatory interest rate i to the final point in time of the investment period.

Final-Value of Assets V_n	The final-value of assets, unlike the amount of capital, splits the calculatory interest rate into the debit interest rate (for the interest on the capital invested in the investment) and the credit interest rate (for the reinvestment of the returns).

B_a	Present value of expenditures/disbursements
B_e	Present value of income/proceeds
A_q	Profit annuity
A_a	Expenditure annuity
A_e	Income annuity
z_k	Payment balance of the k^{th} period with $k = 1,2,\ldots,n$
e_k	Income/proceeds of the k^{th} period with $k = 1,2,\ldots,n$
a_k	Expenditures/disbursements of the k^{th} period with $k = 1,2,\ldots,n$
t_k	Point in time of the k^{th} period with $k = 1,2,\ldots,n$
n	Total period
i	Interest rate, rate of discount
r	Internal rate of return (IRR), internal rate of discount

i_{debit}	Debit interest rate for the return on capital invested
i_{credit}	Credit interest rate for reinvestment of the returns

Payment Series of an Investment

Point in Time	t_0	t_1	t_2	t_3	...	t_{n-1}	t_n
Income	e_0	e_1	e_2	e_3	...	e_{n-1}	e_n
Expenditures	a_0	a_1	a_2	a_3	...	a_{n-1}	a_n
Payment Series	$e_0 - a_0$	$e_1 - a_1$	$e_2 - a_2$	$e_3 - a_3$	...	$e_{n-1} - a_{n-1}$	$e_n - a_n$

Adequate Target Rate

The adequate target rate displays the return of the investment sum. In this way, the real interest due for a (nominal shown) loan can be accounted for. With the help of the interest rate, the payment series of an investment is transformed into an indicator, on the basis of which the advantageousness can be evaluated (comparatively).

The rate of discount:

- is the minimum interest rate used to calculate interest on outstanding amounts;
- is normally above the market interest rate;
- must be set higher, the higher the risk of an investment is;

- is, when using borrowed capital (outside capital) $\geq$ the interest rate of the loan of the borrowed capital by a possible reinvestor.

7.6.2 Fundamentals of Financial Mathematics

Compound Interest Calculation

Accumulation factor $(1+i)^n$

Discount factor $\dfrac{1}{(1+n)^n} = (1+n)^{-n}$

Amount of capital $C_n = C_0 \cdot (1+i)^n$

Net present value (NPV) $C_0 = C_n \cdot \dfrac{1}{(1+i)^n} = C_n \cdot (1+i)^{-n}$

Examples:

Accumulation factor/discount factor:

End of Year	Accumulation Factor 8 %	Discount Factor 8 %
1	1.0800	0.9259
2	1.1664	0.8573
3	1.2597	0.7938
4	1.3605	0.7350

Amount of capital: $C_0 = \$2{,}000;\ i = 0.08;\ n = 5$

$$C_n = \$2{,}000 \cdot (1 + 0.08)^5 = \$2{,}938.66$$

Net present value: $C_n = \$2{,}938.66;\ i = 0.08;\ n = 5$

$$C_0 = \$2{,}938.66 \cdot (1 + 0.08)^{-5} = \$2{,}000$$

Annuity Calculation

Annuity present value factor

$$\frac{(1+i)^n - 1}{(1+i)^n \cdot i}$$

Annuity factor

$$\frac{1}{\text{annuity present value factor}} = \frac{(1+i)^n \cdot i}{(1+i)^n - 1}$$

Actual cash value of an annuity

$$R = C_0 \cdot \frac{1}{\text{annuity present value factor}} = C_0 \cdot \text{annuity factor}$$

$$R = C_0 \cdot \frac{(1+i)^n \cdot i}{(1+i)^n - 1}$$

Present value of an annuity

$$C_0 = R \cdot \frac{(1+i)^n - 1}{(1+i)^n \cdot i}$$

Present value of a perpetuity

$$C_0 = \frac{R}{i}$$

Examples:

Annuity factor/annuity present value factor:

End of Year	**Annuity Factor 10 %**	**Annuity Present Value Factor 10 %**
1	1.1000	0.9091
2	0.5762	1.7355
3	0.4021	2.4869
4	0.3155	3.1699

Actual cash value of an annuity

$$C_0 = \$4,500;\ i = 0.1;\ n = 4$$

$$R = \$4,500 \cdot \frac{(1+0.1)^4 \cdot 0.1}{(1+0.1)^4} - 1 = \$1,419.62$$

Present value of an annuity

$$R = \$1,419.62;\ i = 0.1;\ n = 4$$

$$R = \$1,419.62 \cdot \frac{(1+0.1)^4 - 1}{(1+0.1)^4 \cdot 0.1} = \$4,500$$

Present value of a perpetuity

$$R = \$1,419.62;\ i = 0.1$$

$$C_0 = \frac{\$1,419.75}{0.1} = \$14,196.20$$

7.6.3 Methods of Static Investment Calculation

In the static investment calculation, the reference to time or to temporal (dynamic) changes is either not considered at all or only incompletely.

Cost Comparison Method

For the cost comparison method, the option with the lowest costs is recommended. The procedure is used in practice for replacement and rationalisation investments, since the revenues are not or only subordinately relevant for the decision.

Profit Comparison Method

By definition, the profit comparison method considers not only costs but also revenues (profit $=$ revenues $-$ costs). Here, the option with the highest profit is recommended.

Amortisation Calculation (Pay-back Method, Pay-off Method or Pay-out Method)

The amortisation calculation focuses on the question of when or within which period of time the invested capital of an investment project will be amortised. The project with the shortest amortisation period is recommended.

Profitability Calculation

The decision criterion of an evaluation of investment alternative is the period profitability within a certain period of time. The profitability puts the profit in relation to the capital employed. The investment with the highest period profitability is recommended.

7.6.4 Methods of Dynamic Investment Calculation

The dynamic methods of investment calculation are based on payment series that are related to a specific point in time, i.e. they are based on a specific date. A payment series corresponds to the balances of proceeds and disbursements.

7.6.4.1 Net Present Value Method (Net Present Value, Amount of Capital, Final Asset Value)

With the net present value method, the decision criterion of an investment is the capital value (NPV or amount of capital).

Payment balance $z_k = e_k - a_k$

Net present value
$$C_0 = z_0 + \frac{z_1}{(1+i)} + \frac{z_2}{(1+i)^2} + \ldots + \frac{z_n}{(1+i)^n}$$
$$= z_0 + z_1 \cdot (1+i)^{-1} + z_2 \cdot (1+i)^{-2} + \ldots + z_n \cdot (1+i)^{-n}$$

Amount of capital
$$C_n = z_0 \cdot (1+i)^n + z_1 \cdot (1+i)^{n-1} + \ldots + z_n$$

Final-Value of Assets
$$V_n = z_0 \cdot (1+i_{debit})^n + z_1 \cdot (1+i_{credit})^{n-1} + \\ + z_2 \cdot (1+i_{credit})^{n-2} + \ldots + z_n$$

According to the net present value method, an investment is considered appropriate if the capital value (i.e. the net present value, the amount of capital or the final asset value) is greater than or equal to zero $(C_0; C_n; V_n \geq 0)$.

Capital value > 0 $\Rightarrow$ The investment is considered advantageous compared to an alternative (financial) investment or an expected minimum return.

Capital value $= 0$ $\Rightarrow$ The investment achieves (at least) the required minimum return.

Capital value < 0 $\Rightarrow$ The investment is not appropriate.

Example:

$i = 0.08\ (8\%);\ \ n = 4$ periods with the time points t_0 to t_4

Point in Time	t_0	t_1	t_2	t_3	t_4
Proceeds	$\$0$	$\$100,000$	$\$120,000$	$\$150,000$	$\$45,000$
Disbursements	$\$60,000$	$\$30,000$	$\$50,000$	$\$105,000$	$\$40,000$
Payment Series	$-\$60,000$	$\$70,000$	$\$70,000$	$\$45,000$	$\$5,000$

Net Present Value

$$C_0 = -\$60,000 + \$70,000 \cdot (1.08)^{-1} + \$70,000 \cdot (1.08)^{-2} + \$45,000 \cdot$$

$$\cdot\, (1.08)^{-3} + \ \$5,000 \cdot (1.08)^{-4}$$

$$C_0 = \$104,226.13$$

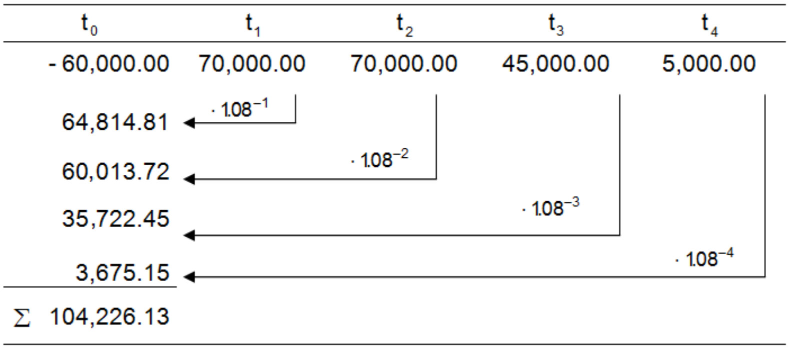
Amount of Capital

$$C_n = -\$60,000 \cdot (1.08)^4 + \$70,000 \cdot (1.08)^3 + \$70,000 \cdot (1.08)^2 + \$45,000 \cdot \\ \cdot (1.08) + \$5,000$$

$$C_n = \$141,798.50$$

The amount of capital corresponds to the net present value of capital, which bears interest (at 8 %) over the entire period:

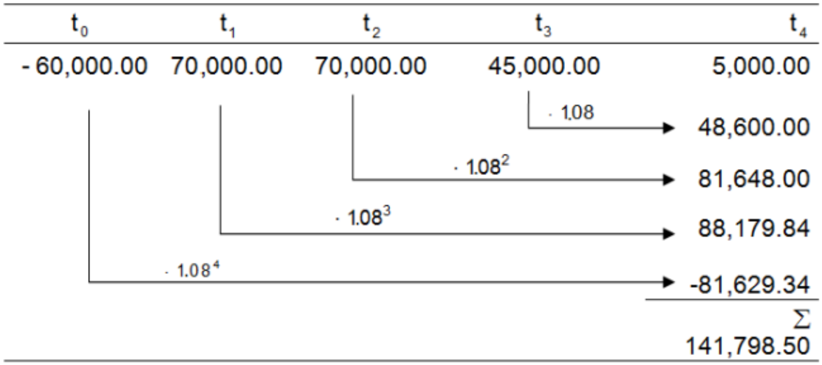
$$C_n = C_0 \cdot q^n$$



$C_n \cdot$ discount factor $= C_0$ $\quad$ $C_0 \cdot$ accumulation factor $= C_n$

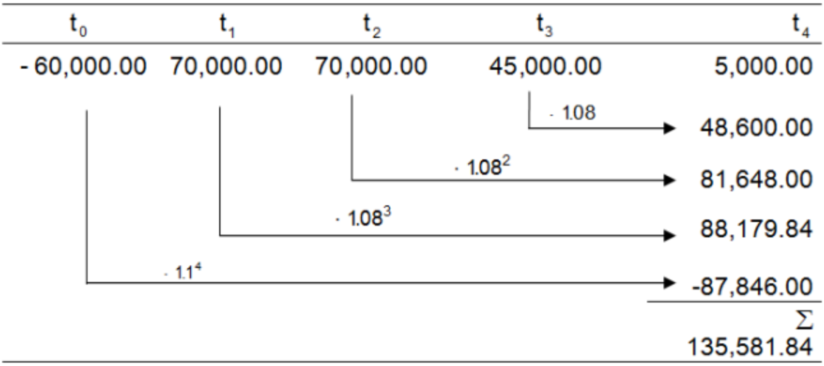
$\Rightarrow \$141,798.50 \cdot 1.08^{-4}$ $\quad$ $\Rightarrow \$104,226.13 \cdot 1.08^{4}$

$= \$104,226.13$ $\quad$ $= \$141,798.50$



Final Asset Value

Debit interest rate $= 10\,\%$ Credit interest rate $= 8\,\%$



Capital values (net present value, amount of capital or final asset value) express the respective monetary value, which results from the assumed or expected incoming and outgoing payments of a time series (investment), calculated at an internal rate of discount, at the beginning or end of a considered period.

7.6.4.2 Annuity Method

The annuity method expresses the (monetary) valuation of an investment in terms of periods. It compares the present value of the income with the present value of the expenditures.

Present value of income
$$B_e = \sum_{k=0}^{n} \frac{e_k}{(1+i)^k}$$

Present value of expenditures
$$B_a = \sum_{k=0}^{n} \frac{a_k}{(1+i)^k}$$

Income annuity $$A_e = B_e \cdot \frac{q^n \cdot (q-1)}{q^n - 1}$$

Expenditure annuity $$A_a = B_a \cdot \frac{q^n \cdot (q-1)}{q^n - 1}$$

Profit annuity[25] $$A_g = A_e - A_a$$

$$\text{or } A_g = \underbrace{C_0}_{\text{NPV}} \cdot \underbrace{\frac{(1+i)^n \cdot i}{(1+i)^n - 1}}_{\substack{\text{annuity factor} \\ \text{in arrears}}}$$

If the profit annuity A_g is greater than or equal to zero, the investment is advantageous:
$A_g = A_e - A_a \geq 0$

Profit annuity > 0 ⇒ The investment is considered advantageous compared to an alternative (financial) investment or expected minimum return.

Profit annuity $= 0$ ⇒ The investment achieves (at least) the required minimum return.

Profit annuity < 0 ⇒ The investment is not appropriate.

<u>Example:</u>

$i = 0.08\ (8\%)$; $n = 4$ periods with the time points t_0 to t_4

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[25] Profit in the sense of surplus revenue.

Point in Time	t_0	t_1	t_2	t_3	t_4
Proceeds	$\$0$	$\$100,000$	$\$120,000$	$\$150,000$	$\$45,000$
Disbursements	$\$60,000$	$\$30,000$	$\$50,000$	$\$105,000$	$\$40,000$
Payment Series	$-\$60,000$	$\$70,000$	$\$70,000$	$\$45,000$	$\$5,000$

$$B_a = \$60,000 + \frac{\$30,000}{1.08} + \frac{\$50,000}{1.08^2} + \frac{\$105,000}{1.08^3} + \frac{\$40,000}{1.08^4} = \$243,398.30$$

$$B_e = \frac{\$100,000}{1.08} + \frac{\$120,000}{1.08^2} + \frac{\$150,000}{1.08^3} + \frac{\$45,000}{1.08^4} = \$347,624.43$$

$$A_a = \$243,398.30 \cdot \frac{0.08 \cdot (1.08)^4}{(1.08)^4 - 1} = \$73,487.01$$

$$A_e = \$347,624.43 \cdot \frac{0.08 \cdot (1.08)^4}{(1.08)^4 - 1} = \$104,955.05$$

$$A_g = \$104,995.05 - \$73,487.01 = \$31,468.04$$

or alternatively with $A_g = C_0 \cdot \frac{(1+i)^n \cdot i}{(1+i)^n - 1}$

$$C_0 = -\$60,000 + \$70,000 \cdot (1.08)^{-1} + \$70,000 \cdot (1.08)^{-2} + \$45,000 \cdot \cdot (1.08)^{-3} + \$5,000 \cdot (1.08)^{-4}$$

$$C_0 = \$104,226.13$$

$$\Rightarrow A_g = \$104,226.13 \cdot \frac{(1+0.08)^4 \cdot 0.08}{(1+0.08)^4 - 1} = \$31,468.04$$

The profit annuity of \$31,468.04 corresponds to the annual surplus (average annual profit rate) of the investment at an interest rate of $8\,\%$. The investment thus appears to be economically reasonable due to the positive profit annuity.

7.6.4.3 Internal Rate of Return Method

The internal rate of return r is determined by an approximation method and compared with the rate of discount i.

Discount Factor for the k^th Year

$$q^{-k} = (1+r)^{-k}$$

Equation for Calculating the Internal Rate of Return

$$0 = -z_0 + \frac{z_1}{(1+r)^1} + \frac{z_2}{(1+r)^2} + \frac{z_3}{(1+r)^3} + \ldots + \frac{z_n}{(1+r)^n}$$

$$\Rightarrow 0 = -z_0 + \sum_{k=1}^{n} \frac{z_k}{(1+r)^k}$$

Examples:

(1) One Period Case

Point in Time	t_0	t_1
Payment Series	$-\$100$	$\$110$

$\Rightarrow$ Internal rate of return: $0 = -100 + \dfrac{110}{(1+r)^1} \Rightarrow r = 0.1\,(10\,\%)$

$\Rightarrow \dfrac{\text{profit}}{\text{capital}} = \dfrac{110-100}{100} = 10\,\%$ profit

(2) Payment Series Over Several Periods

Point in Time	t_0	t_1	t_2
Proceeds	$\$0$	$\$82,100$	$\$73,000$
Disbursements	$\$50,000$	$\$55,000$	$\$43,000$

Setting up the equation to calculate r:

$$0 = -\$50,000 + \frac{\$82,100 - \$55,000}{q} + \frac{\$73,000 - \$43,000}{q^2}$$

Multiplication of the equation by $\dfrac{q^2}{1,000}$:

$$0 = \left(-\$50,000 + \frac{\$27,100}{q} + \frac{\$30,000}{q^2}\right) \cdot \frac{q^2}{1,000}$$

$$0 = \left(-\$50,000 \cdot \frac{q^2}{1,000}\right) + \left(\frac{\$27,100}{q} \cdot \frac{q^2}{1,000}\right) + \left(\frac{\$30,000}{q^2} \cdot \frac{q^2}{1,000}\right)$$

$$0 = -50\,q^2 + 27.1\,q + 30 \qquad \mid \div (-50)$$

Application of the p/q formula:

$$0 = q^2 - 0.542q - 0.6 \qquad \Rightarrow q_{1/2} = -\frac{(-0.542)}{2} \pm \sqrt{\left(\frac{-0.542}{2}\right)^2 - (-0.6)}$$

$$\Leftrightarrow q_{1/2} = 0.271 \pm \sqrt{0.073441}$$

$$\Rightarrow q_1 = 0.271 + 0.8206 = 1.0916$$

$$\Rightarrow q_2 = 0.271 - 0.8206 = -0.5496$$

q_1 corresponds to an internal rate of return of $r \approx 0.0916\,(9.16\,\%)$. If this internal rate of return is above the (assumed) discount rate, the investment is advantageous.

No internal interest rate can be assigned to q_2 because of the negative sign. No interest rate could be assigned to a $q-$value below 1 either.

Chapter 8

Optimisation of Linear Models

By means of the *Lagrange method* or *linear optimisation*, the relative extremes (minima or maxima) of a linear (target) function can be determined under restrictive linear constraints.

If the constraints are given in the form of an equation, the model can be solved with the help of the *Lagrange method*. However, if the constraints consist of inequalities, the model can be solvable using a *linear programming approach (linear programming, linear optimisation)*.

8.1 Lagrange Method

8.1.1 Introduction

The method of Lagrange multipliers[1] is a mathematical procedure that determines the relative extremes of a linear mathematical model (= linear target function and linear constraints) if the constraints are given in the form of *equations*.

8.1.2 Formation of the Lagrange Function

Given is a (target) function

$$f = f(x_1, x_2, \ldots, x_n) \qquad x_i > 0 \quad \text{with} \quad i = 1, \ldots, n$$

for which the local extremes are to be determined.

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[1] Joseph-Louis Lagrange (1736 - 1813) was an Italian mathematician.

F. W. Peren, *Math for Business and Economics*,
https://doi.org/10.1007/978-3-662-68860-1_8

The function f is limited by the constraints

$$\phi = \phi_j(x_1, x_2, \ldots, x_n) \qquad j = 1, ..., m$$

The Lagrange function $L = L(x_1, x_2, \ldots x_n)$ additively links the (target) function with the restrictions:

$$\begin{aligned} L = \quad & f(x_1, \ldots, x_n) \\ & + \lambda_1 \phi_1(x_1, \ldots, x_n) + \\ & + \lambda_2 \phi_2(x_1, \ldots, x_n) + \\ & \vdots \\ & + \lambda_m \phi_m(x_1, \ldots, x_n) \end{aligned}$$

λ_j: Lagrange multiplier for the j^{th} constraint $\quad \lambda_j \in \mathbb{R}$

$j = 1, \ldots, m$

$m < n$

8.1.3 Determination of the Solution

According to the necessary conditions for determining relative extremes, the first partial derivatives of the Lagrange function are set to zero after the variables x_i with $i = 1, ..., n$, and after the Lagrange multipliers λ_j with $j = 1, ..., m$. The result is a system of $(n+m)$ equations with $(n+m)$ unknowns and thus a unique solution:

$$\frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial x_2} = \ldots = \frac{\partial L}{\partial x_n} = \frac{\partial L}{\partial \lambda_1} = \frac{\partial L}{\partial \lambda_2} = \ldots = \frac{\partial L}{\partial \lambda_m} = 0$$

$\Rightarrow (n+m)$ equations with $(n+m)$ unknowns.

From this system of equations, the solutions of the variables and thus also the coordinates of the searched extrema are uniquely determined.

Example:

There are two different drinks to choose from. The total utility from consuming these drinks should be maximised with a limited budget. The budget is $\$60$ and the utility is presented with the following function:

$$U(x,y) = 2xy$$

The budget restriction can be formulated by the prices of goods as an equation. One unit x (wine) costs $\$5$ and one unit y (water) costs $\$1$.

$$\begin{aligned} 5x+y &= 60 \\ 60-5x-y &= 0 \qquad \text{(positive absolute term)} \end{aligned}$$

Note: If the restriction is transformed so that the absolute term of this equation is positive, λ can be directly interpreted hereafter.

Target function: $U(x,y) = 2xy$

Constraint: $60-5x-y = 0$

$$(1) \quad L(x,y,\lambda) = \underbrace{2xy}_{\substack{\text{target}\\\text{function}}} + \underbrace{\lambda \cdot (60-5x-y)}_{\text{constraint}}$$

(2) Necessary constraints for local extremes:

$$L'x = 2y - 5\lambda = 0 \qquad \text{(I)} \qquad \Rightarrow \qquad 2y - 5\lambda = 0$$

$$L'y = 2x - \lambda = 0 \qquad \text{(II)} \qquad \Rightarrow \qquad 2x - \lambda = 0$$

$$L'\lambda = 60 - 5x - y = 0 \qquad \text{(III)} \qquad \Rightarrow \qquad 60 - 5x - y = 0$$

$\Rightarrow$ 3 equations with 3 unknowns

Solution of the equation system:

(I) and (II) isolate λ:

(I) $\lambda = \frac{2}{5}y$

(II) $\lambda = 2x$

(I) and (II) are equated to each other:

$$\frac{2}{5}y = 2x \quad \Leftrightarrow \quad x = \frac{1}{5}y$$

Insert x in (III):

$$60 - 5 \cdot \frac{1}{5}y - y = 0$$

$$y = 30$$

$$\Rightarrow x = 6; \quad \lambda = +12$$

The maximum utility is reached with a given utility function (= functionalisation of the utility) and a limited budget of $\$60$ within the considered time, if $x = 6$ units wine and $y = 30$ units water are consumed.

The maximum utility that is to be generated is defined as

$$U_{max} \;=\; 2 \cdot 6 \cdot 30 \;=\; 360 \text{ units (utility units)}$$

With full use of the budget $\$60$, no alternative x/y-combination can be found which could provide a greater benefit than the local maximum identified here.

8.1.4 Interpretation of λ

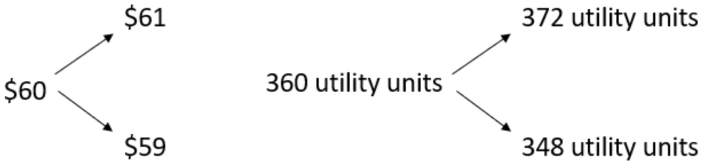
If the default j^{th} restriction is incremented/decremented by one unit, the result of the target function varies approximately by λ_j units. The following table gives an overview of the changes in the results of the target function.

λ Δ restriction	positive	negative
positive	increment	decrement
negative	decrement	increment

Interpretation of λ from the example of chapter 8.1.3:

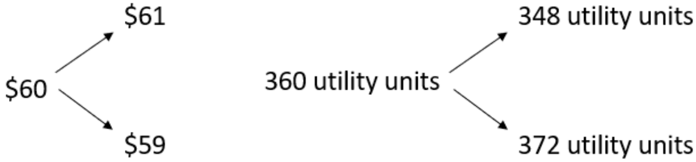
$$\lambda \;=\; +12$$

<u>Note:</u> λ ist nondimensional



If the budget of currently $\$60$ was increased ceteris paribus (c.p.) by $\$1$ to $\$61$, the maximum utility would go up absolutely by 12 utility units to 372 utility units. If instead the budget c.p. was reduced by $\$1$ to $\$59$, the maximum utility would decrease by 12 utility units to 348 utility units.

If $\lambda = -12$, then increasing the budget from current $\$61$ c.p. by $\$1$ to $\$61$, would decrease the maximum utility absolutely by 12 utility units to 348 utility units. If instead the budget c.p. were reduced by $\$1$ to $\$59$, the maximum utility to be achieved would then increase by 12 utility units to 372 utility units.



8.1.5 Identification of the Type of Optimum

In the classical solution of a Lagrangian problem, as also shown in chapter 8.1.3, it is not yet clear what kind of extremum (minimum or maximum) is considered.

There are different solutions to this problem, which will be shown in the following by three examples.

Example 1:

Target function: $f(x,y) = 4 - x^2 - y^2$

Constraint: $g(x,y) = x + y = 1$

Step 1: Convert the constraint to 0

$\Rightarrow \qquad x + y = 1 \;\Leftrightarrow\; 1 - x - y = 0$

Step 2: Form Lagrange function

$\Rightarrow \quad L(x,y,\lambda) = f(x,y) + \lambda \cdot g(x,y) \quad \rightarrow$ The constraint is multiplied by the variable.

$$L(x,y,\lambda) = 4 - x^2 - y^2 + \lambda(1 - x - y)$$

$$L(x,y,\lambda) = 4 - x^2 - y^2 + \lambda - \lambda x - \lambda y$$

Step 3: Building of all partial derivatives 1st order according to each independent variable x, y and λ

$$\Rightarrow \quad L'_x = -2x - \lambda$$

$$L'_y = -2y - \lambda$$

$$L'_\lambda = 1 - x - y$$

Step 4: Necessary condition $\rightarrow$ set all first order derivatives equal to zero

$$\Rightarrow \quad -2x - \lambda = 0$$

$$-2y - \lambda = 0$$

$$1 - x - y = 0$$

Step 5: Solve the equation system

(I) $-2x - \lambda = 0 \mid +\lambda \;\Rightarrow\; -2x = \lambda \mid : (-2) \;\Rightarrow\; x = -\frac{1}{2}\lambda$

(II) $-2y - \lambda = 0 \mid +\lambda \;\Rightarrow\; -2y = \lambda \mid : (-2) \;\Rightarrow\; y = -\frac{1}{2}\lambda$

(III) $1 - x - y = 0$

insert x and y into equation (III)

(III) $1 + \frac{1}{2}\lambda + \frac{1}{2}\lambda = 0 \;\Leftrightarrow\; 1 + \lambda = 0 \;\Leftrightarrow\; \lambda = -1$

$\Rightarrow\; x = -\frac{1}{2}\cdot(-1) = \frac{1}{2} \qquad y = -\frac{1}{2}\cdot(-1) = \frac{1}{2}$

Solution: There could be an extreme point at $P(\frac{1}{2} \mid \frac{1}{2})$.

Step 6: Building of all partial derivatives 2nd order to identify the type of the possible extreme point

Partial derivatives 1st order

$L'_x = -2x - \lambda \quad L'_y = -2y - \lambda \quad L'_\lambda = 1 - x - y$

Partial derivatives 2nd order

$L''_{xx} = -2 \quad L''_{xy} = 0 \quad L''_{x\lambda} = -1$

$L''_{yx} = 0 \quad L''_{yy} = -2 \quad L''_{y\lambda} = -1$

$L''_{\lambda x} = -1 \quad L''_{\lambda y} = -1 \quad L''_{\lambda\lambda} = 0$

Step 7: Identifying the type of extreme point

$\Rightarrow$ Set up Hessian matrix with λ as the first variable and using **Sarrus’ Rule** for a $3x3$ matrix.

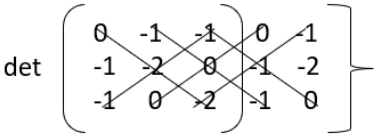
$$\begin{pmatrix} L''_{\lambda\lambda} & L''_{\lambda x} & L''_{\lambda y} \\ L''_{x\lambda} & L''_{xx} & L''_{xy} \\ L''_{y\lambda} & L''_{yx} & L''_{yy} \end{pmatrix}$$

Insert the concrete values of the derivatives of second order

$$\Rightarrow \begin{pmatrix} 0 & -1 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}$$

If there were variables here, for $x = \frac{1}{2}$ and for $y = \frac{1}{2}$ would be used. All values inside the Hessian matrix are then numerical (without variables).

Step 8: Calculate the determinant



→ add the first two columns

→ multiply the numbers on the respective diagonal by each other

$$\text{det} = 0 \cdot (-2) \cdot (-2) + (-1) \cdot 0 \cdot (-1) + (-1) \cdot (-1) \cdot 0 - (-1) \cdot (-2) \cdot (-1) - \\ - 0 \cdot 0 \cdot 0 - (-1) \cdot (-1) \cdot (-2) = 0 + 0 + 0 + 2 - 0 + 2 = 4$$

Solution: $4 > 0$, which means that there is an extremum at the point $P(0.5\,|\,0.5)$.

Since $L''_{xx} = -2 < 0$ and $L''_{yy} = -2 < 0$, the function is concavely curved in both directions, so that there is a maximum at the point $P(0.5\,|\,0.5)$.

Alternative solution for identifying the type of optimum

The identification of the type of a possible extremum can also be done by inserting **adjacent values** of the function value of the extremum under the constraint $x + y = 1$.

$$f(x_0, y_0) = f(\frac{1}{2}; \frac{1}{2}) = 4 - (\frac{1}{2})^2 - (\frac{1}{2})^2 = 4 - \frac{1}{4} - \frac{1}{4} = 3.5$$

Adjacent values (exemplary selection):

$$f(\frac{1}{3}; \frac{2}{3}) = 4 - (\frac{1}{3})^2 - (\frac{2}{3})^2 = 3.44 < 3.5$$

$$f(\frac{4}{5}; \frac{1}{5}) = 4 - (\frac{4}{5})^2 - (\frac{1}{5})^2 = 3.32 < 3.5$$

$\Rightarrow$ Maximum at the point $(\frac{1}{2}\,|\,\frac{1}{2})$

Example 2:

Calculation of possible extrema with three unknown variables under two constraints using the **Sarrus' Rule**.

Searched are the extrema of the function $f(x,y,z)$ under the constraints $g_1(x,y,z)$ and $g_2(x,y,z)$.

$$f(x,y,z) = x^2 + 3y^2 + 2z^2$$

$$g_1(x,y,z) = 4x + 12y = 120$$

$$g_2(x,y,z) = 6y + 12z = 120$$

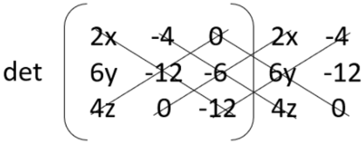
Step 1: Set constraints $= 0$

$$g_1(x,y,z) = 120 - 4x - 12y = 0$$

$$g_2(x,y,z) = 120 - 6y - 12z = 0$$

Step 2: Calculate the determinant

$$\Rightarrow \begin{pmatrix} f'(x) & g_1'(x) & g_2'(x) \\ f'(y) & g_1'(y) & g_2'(y) \\ f'(z) & g_1'(z) & g_2'(z) \end{pmatrix} = 0$$



$$\det \left(\begin{array}{ccc|cc} 2x & -4 & 0 & 2x & -4 \\ 6y & -12 & -6 & 6y & -12 \\ 4z & 0 & -12 & 4z & 0 \end{array} \right)$$

$$\det = 2x \cdot (-12) \cdot (-12) + (-4) \cdot (-6) \cdot 4z + 0 \cdot 6y \cdot 0 - 0 \cdot (-12) \cdot 4z - \\ - 2x \cdot (-6) \cdot 0 - (-4) \cdot 6y \cdot (-12) = 288x + 96z - 288y$$

Set zero: $288x + 96z - 288y = 0 \,|\, : 96$

$$3x + z - 3y = 0$$

Solved for x,y,z: $x = -\frac{1}{3}z + y$

$$y = x + \frac{1}{3}z$$

$$z = -3x + 3y$$

Step 3: Insert in constraint to calculate x, y and z

$g_1(x,y,z) = 120 - 4x - 12y = 0$ with $y = x + \frac{1}{3}z$

$$\Rightarrow \quad 120 - 4x - 12(x + \frac{1}{3}z) = 0$$

$$\Leftrightarrow \quad 120 - 4x - 12x - 4z = 0$$

$$\Leftrightarrow \quad 120 - 16x - 4z = 0$$

$$\Leftrightarrow \quad 16x = 120 - 4z$$

$x = 7.5 - \frac{1}{4}z$ and $y = x + \frac{1}{3}z$ (see above)

⇒ insert into the second constraint:

$$g_2(x,y,z) = 120 - 6y - 12z = 0$$

$$\Rightarrow \quad 120 - 6(x + \frac{1}{3}z) - 12z = 0$$

$$\Rightarrow \quad 120 - 6x - 2z - 12z = 0$$

$$\Rightarrow \quad 120 - 6(7,5 - \frac{1}{4}z) - 14z = 0$$

$$\Leftrightarrow \quad 120 - 45 + \frac{3}{2}z - 14z = 0$$

$$12.5z = 75$$

$$z = 6$$

$$x = 7.5 - \frac{1}{4}z = 7.5 - \frac{1}{4} \cdot 6 = 6$$

$$y = x + \frac{1}{3}z = 6 + \frac{1}{3} \cdot 6 = 8$$

Possible extremum at the point $(6|8|6)$.

Step 4: Determination of the type of extremum

Since $f''_{xx} = 2 > 0$, $f''_{yy} = 6 > 0$ and $f''_{zz} = 4 > 0$ the function is convexly curved in all three directions, so there is a minimum here.

Step 5: Determination of lambda

$$L(x,y,z,\lambda_1,\lambda_2) = x^2 + 3y^2 + 2z^2 + \lambda_1(120 - 4x - 12y) + \lambda_2(120 - 6y - 12z)$$

$$L'_x = 2x - 4\lambda_1 = 0 \qquad\qquad L'_y = 6y - 12\lambda_1 - 6\lambda_2 = 0$$

$$\Rightarrow \quad 4\lambda_1 = 2x \qquad\qquad \Rightarrow \quad 6\lambda_2 = 6y - 12 \cdot 3$$

$$\Rightarrow \quad \lambda_1 = \frac{2}{4} \cdot 6 = 3 \qquad\qquad \Rightarrow \quad \lambda_2 = \frac{6 \cdot 8 - 12 \cdot 3}{6} = 2$$

Example 3:

Determination of the extreme points with three unknown variables under two constraints.

Searched are the extreme points of the function $f(x_1,x_2,x_3)$ under the constraints $g_1(x_1,x_2,x_3)$ and $g_2(x_1,x_2,x_3)$.

$$f(x_1,x_2,x_3) = (x_1 - 2)^2 + (x_2 - 3)^2 - x_3^2 = x_1^2 - 4x_1 + 4 + x_2^2 - 6x_2 + 9 - x_3^2$$

$$g_1(x_1,x_2,x_3) = x_1 + x_2 + x_3 = 2$$

$$g_2(x_1,x_2,x_3) = 3x_1 + x_2 - x_3 = 2$$

Step 1: Set constraints $= 0$

$$g_1(x_1,x_2,x_3) = 2 - x_1 - x_2 - x_3 = 0$$

$$g_2(x_1,x_2,x_3) = 2 - 3x_1 - x_2 + x_3 = 0$$

Step 2: Form Lagrange function

$$L(x_1,x_2,x_3,\lambda_1,\lambda_2) = (x_1-2)^2+(x_2-3)^2-x_3^2+\lambda_1(2-x_1-x_2-x_3=0)+ \\ +\lambda_2(2-3x_1-x_2+x_3)$$

Step 3: Set all partial derivatives 1st order $= 0$ and convert to x_1, x_2 and x_3

$$L'_{x_1} = 2\cdot(x_1-2)-\lambda_1-3\lambda_2 = 0$$

$$L'_{x_2} = 2\cdot(x_2-3)-\lambda_1-\lambda_2 = 0$$

$$L'_{x_3} = -2x_3-\lambda_1+\lambda_2 = 0$$

$$L'_{\lambda_1} = 2-x_1-x_2-x_3 = 0$$

$$L'_{\lambda_2} = 2-3x_1-x_2+x_3 = 0$$

$$x_1 = 2+0.5\lambda_1+1.5\lambda_2$$

$$x_2 = 3+0.5\lambda_1+0.5\lambda_2$$

$$x_3 = -0.5\lambda_1+0.5\lambda_2$$

Step 4: Solve the equation system

$$L'_{\lambda_1} = 2-x_1-x_2-x_3 = 0$$

$$L'_{\lambda_1} = 2-(2+0.5\lambda_1+1.5\lambda_2)-(3+0.5\lambda_1+0.5\lambda_2)-(-0.5\lambda_1+0.5\lambda_2) = 0$$

$$-3-0.5\lambda_1-2.5\lambda_2 = 0$$

$$\lambda_1 = -6-5\lambda_2$$

insert into the derivatives converted to x_1, x_2 and x_3

$$x_1 = 2+0.5\lambda_1+1.5\lambda_2$$

$$x_1 = 2+0.5\cdot(-6-5\lambda_2)+1.5\lambda_2$$

$$x_1 = -1-\lambda_2$$

$x_2 = 3 + 0.5\lambda_1 + 0.5\lambda_2$

$x_2 = 3 + 0.5 \cdot (-6 - 5\lambda_2) + 0.5\lambda_2$

$x_2 = -2\lambda_2$

$x_3 = -0.5\lambda_1 + 0.5\lambda_2$

$x_3 = -0.5 \cdot (-6 - 5\lambda_2) + 0.5\lambda_2$

$x_3 = 3 + 3\lambda_2$

insert into L'_{λ_2}

$L'_{\lambda_2} = 2 - 3x_1 - x_2 + x_3 = 0$

$2 - 3 \cdot (-1 - \lambda_2) - (-2\lambda_2) + (3 + 3\lambda_2) = 0$

$8 + 8\lambda_2 = 0$

$\lambda_2 = -1$

$\lambda_1 = -6 - 5\lambda_2 =$

$= \lambda_1 = -6 - 5 \cdot (-1) =$

$= \lambda_1 = -6 + 5 = -1$

$\lambda_1 = -1$

$x_1 = -1 - \lambda_2 = -1 - (-1)$

$x_1 = 0$

$x_2 = -2\lambda_2 = -2 \cdot (-1)$

$x_2 = 2$

$x_3 = 3 + 3\lambda_2 = 3 + 3 \cdot (-1)$

$x_3 = 0$

Possible extremum at the point $(2|0|2)$

Interpretation of lambda: If both constraints ceteris paribus, i.e. under otherwise equal conditions, are each increased or decreased by one unit, the result of the function $f(x_1,x_2,x_3)$ decreases or increases by 1 unit.

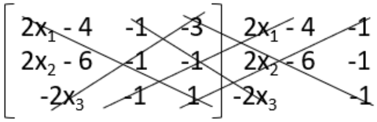
Determination of the type of extremum using the **Sarrus' Rule**.

$$g_1(x_1,x_2,x_3) = 2 - x_1 - x_2 - x_3$$

$$g_2(x_1,x_2,x_3) = 2 - 3x_1 - x_2 + x_3$$

$$\Rightarrow \begin{pmatrix} f'(x_1) & g_1'(x_1) & g_2'(x_1) \\ f'(x_2) & g_1'(x_2) & g_2'(x_2) \\ f'(x_3) & g_1'(x_3) & g_2'(x_3) \end{pmatrix} = 0$$

det



$$\left[\begin{array}{ccc|ccc} 2x_1 - 4 & -1 & -3 & 2x_1 - 4 & -1 \\ 2x_2 - 6 & -1 & -1 & 2x_2 - 6 & -1 \\ -2x_3 & -1 & 1 & -2x_3 & -1 \end{array}\right]$$

$$\text{det} = (2x_1-4)\cdot(-1)\cdot 1 + (-1)\cdot(-1)\cdot(-2x_3) + (-3)\cdot(2x_2-6)\cdot(-1) - \\ -(-3)\cdot(-1)\cdot(-2x_3) - (2x_1-4)\cdot(-1)\cdot(-1) - (-1)\cdot(2x_2-6)\cdot 1$$

Set zero: $-2x_1 + 4 - 2x_3 + 6x_2 - 18 + 6x_3 - 2x_1 + 4 + 2x_2 - 6$

$$\Rightarrow \quad -4x_1 + 8x_2 + 4x_3 - 16 = 0 \mid :4$$

$$\Leftrightarrow \quad -x_1 + 2x_2 + x_3 - 4 = 0$$

Solved for x_1, x_2, x_3: $x_1 = x_3 + 2x_2 - 4$

$$x_2 = -0.5x_1 - 0.5x_3 + 2$$

$$x_3 = x_1 - 2x_2 + 4$$

Insert into constraints to calculate x_1, x_2, and x_3

$g_1 = (x_1, x_2, x_3) = 2 - x_1 + x_2 + x_3 = 0$ with $x_1 = x_3 + 2x_2 - 4$

$$\Rightarrow \quad 2 - x_3 - 2x_2 + 4 - x_2 - x_3 = 0$$

$$\Leftrightarrow \quad 6 - 3x_2 - 2x_3 = 0$$

$$\Leftrightarrow \quad 3x_2 = 6 - 2x_3$$

$$\Leftrightarrow \quad x_2 = 2 - \frac{2}{3}x_3$$

and $x_3 = x_1 - 2x_2 + 4$ (see above)

$\Rightarrow$ insert into second constraint:

$$g_2(x_1, x_2, x_3) = 2 - 3x_1 - x_2 + x_3 = 0$$

$$\Rightarrow \quad 2 - 3x_1 - (2 - \frac{2}{3}x_3) + x_3 = 0$$

$$\Leftrightarrow \quad 2 - 3x_1 - 2 + \frac{2}{3}x_3 + x_3 = 0$$

$$\Leftrightarrow \quad -3x_1 + \frac{5}{3}x_3 = 0$$

$$\Rightarrow \quad -3 \cdot (x_3 + 2x_2 - 4) + \frac{5}{3}x_3 = 0$$

$$\Leftrightarrow \quad -3x_3 - 6x_2 + 12 + \frac{5}{3}x_3 = 0$$

$$\Rightarrow \quad -3x_3 - 6 \cdot (2 - \frac{2}{3}x_3) + 12 + \frac{5}{3}x_3 = 0$$

$$\Leftrightarrow \quad -3x_3 - 12 + \frac{12}{3}x_3 + 12 + \frac{5}{3}x_3 = 0$$

$$\Leftrightarrow \quad \frac{8}{3}x_3 = 0$$

$$\Leftrightarrow \quad x_3 = 0$$

$$\Rightarrow \quad x_2 = 2 - \frac{2}{3}x_3 = 2$$

$\Rightarrow \quad x_1 = x_3 + 2x_2 - 4 = 0 + 2 \cdot 2 - 4 = 0$

$\Rightarrow$ possible extreme at the point $(0|2|0)$
($=$ same result as via Lagrange)

Determination of the type of extremum

Since $f''_{x_1x_1} = 2 > 0$, $f''_{x_2x_2} = 2 > 0$ but $f''_{x_3x_3} = -2 < 0$, indifference prevails. In x_1 and x_2 directions, the function is convex and in x_3 direction the function is concave curved.

8.2 Linear Optimisation

8.2.1 Introduction

Linear optimisation or *linear programming* (linear planning) is to be used for the determination of extreme values if the constraints of a linear mathematical model are in the form of inequalities and/or equations.

8.2.2 The Linear Programming Approach

(1) Target function

$z = z(x_1, x_2, \ldots, x_n) \Rightarrow$ opt!

opt = optimisation = maximisation or minimisation

(2) Constraints

$\phi_j = \phi_j(x_1, x_2, \ldots, x_n) \leq c_j$ with $j = 1, \ldots, m$

Note: The $\leq$ - restrictions can also be represented as $\geq$ - restrictions when multiplied with -1 and vice versa.

(3) Non-negativity conditions

$x_i \geq 0$ with $i = 1, \ldots, n$

8.2.3 Graphical Solution

For clearer demonstration, the model discussed below will first be limited to two variables, x_1 and x_2, and two restrictions.

The linear programming approach is demonstrated below:

(1) Target function

$z = z(x_1, x_2, \ldots, x_n) \Rightarrow$ opt!

(2) Constraints

$a_{11}x_1 + a_{12}x_2 \leq a_1$ $a_{ij} \in \mathbb{R};\ \ i;j = 1;2 = constant$

$a_{22}x_1 + a_{22}x_2 \leq a_2$

(3) Non-negativity condition

$x_1, x_2 \geq 0$

The solution set is the set of all ordered pairs of variates $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$ that satisfy the above-mentioned constraints:

$$\mathbb{S}_\mathsf{S} = \{(x_1, x_2) \mid (x_1, x_2) \in \mathbb{R} \times \mathbb{R} \ \wedge \ a_{11}x_1 + a_{12}x_2 \leq a_1 \ \wedge \ a_{21}x_1 + a_{22}x_2 \leq a_2\}$$

Each restriction divides the coordinate system into the relevant half-plane. Due to the non-negativity condition, only the fourth quadrant of the coordinate system is relevant. The possible solutions are shown in the following example by the hatched area. The *target function* marks the corresponding straight line (contour line) $z = z(x_1, x_2)$. The optimum (= the searched value pair), (x_1^{opt}, x_2^{opt}), is obtained, depending on the task, by parallel shifting the target function $z = z(x_1, x_2)$. In the case of maximisation (minimisation), the target function is shifted outwards (inwards) - away from (towards) the origin - until it reaches the limited plane due to the restrictions (hatched area).

The optimum is unique if the target function meets a corner of the relevant area. The solution is ambiguous if the target function is parallel (congruent) to one of the constraint lines.

Example:

Two products are manufactured with three different machines, whereby the capacities of the machines are limited. x_1 has a contribution margin of $\$150$ per unit. x_2 has a contribution margin of $\$100$ per unit. The objective is to maximise the contribution margin (CM), under consideration of the limited machines capacities.

Decision variables: $x_1, x_2 =$ quantities of products produced

Target function: $CM(x_1, x_2) = 150x_1 + 100x_2 \Rightarrow$ max!

Constraints: $(1)\ \ 4x_1 + 2x_2 \leq 200$ with $x_1, x_2 \leq 0$

$(2)\ \ 2x_1 + 4x_2 \leq 200$

$(3)\ \ 2x_1 + 2x_2 \leq 120$

Interpretation of the constraint (1):

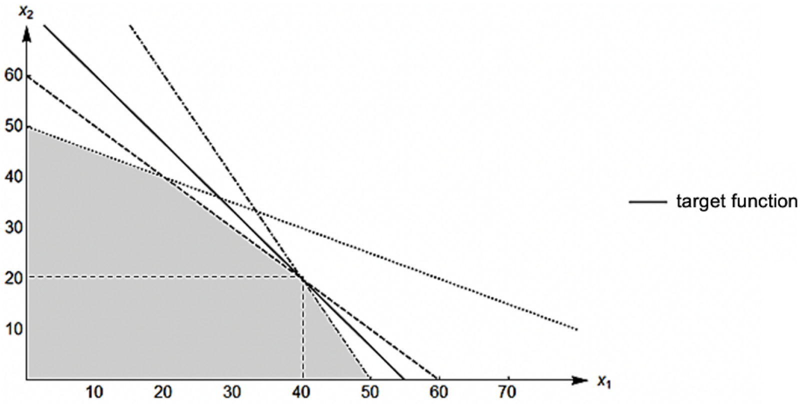
4 TU/QU $\cdot x_1$ QU $+ 2$ TU/QU $\cdot x_2$ QU ≤ 200 TU

QU = quantity units

TU = time units

with $4;2 \;\widehat{=}\;$ production coefficients

$200 \;\widehat{=}\;$ maximum capacity of the first machine



Calculation of the axis intercepts:

(1) $4x_1 + 2x_2 \leq 200$ TU

$x_1 = 0 \Rightarrow x_2^{max} = 100$ QU

$x_2 = 0 \Rightarrow x_1^{max} = 50$ QU

(2) $2x_1 + 4x_2 \leq 200$ TU

$x_1 = 0 \Rightarrow x_2^{max} = 50$ QU

$x_2 = 0 \Rightarrow x_1^{max} = 100$ QU

$(3)\ \ 2x_1 + 2x_2 \leq 120$ TU

$x_1 \ = 0 \ \Rightarrow \ x_2^{max} \ = 60$ QU

$x_2 \ = 0 \ \Rightarrow \ x_1^{max} \ = 60$ QU

target function $150x_1 + 100x_2 =$ e.g. $\$1,500$

$\Rightarrow \ x_1^{max} = 10$ QU

$\Rightarrow \ x_2^{max} = 15$ QU

$\Rightarrow$ Shift of this straight line until it contacts the boundary surface defined by the constraints.

The optimal distribution of the goods x_1 and x_2 can now be observed on the axes. To maximise the total contribution margin, 40 units for x_1 and 20 units for x_2 are produced.

The following solution is obtained for the target function:

$150x_1 + 100x_2$ $\Rightarrow$ max!

$150 \cdot 40 + 100 \cdot 20$ $= \$8,000$

If the optimal quantities of both products are selected, the amount covered is $\$8,000$.

8.2.4 Primal Simplex Algorithm

The *primal (= original) simplex algorithm* is an *iteration method* for the gradual approach to the optimum. It is valid for at least two variables. The maximum value (or minimum value) z_{opt} of the target function $z = z(x_i)$ with $i = 1, \ldots, n,$ is given if the underlying *simplex tableau* is in canonical form and the coefficients of all non-basic variables ≥ 0 (≤ 0). The form of a linear mathematical model is canonical if one variable (=

base variable) has the coefficient 1 in each constraint and this variable does not arise in all other constraints.

8.2.5 Simplex Tableau (Basic Structure)

The simplex tableau is based on the following (basic) linear programming approach:

Primal Simplex Algorithm | Linear Programming Approach

(1) Target function

$z = z(x_i) = b_1 x_1 + b_2 x_2, \dots, + b_n x_n \quad \Rightarrow$ max! or min!

$b_i \in \mathbb{R} = constant$

$i = 1, \dots, n$

(2) Constraints

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq a_1$ $a_{ij} \in \mathbb{R} = constant$

$\vdots$ with $j = 1, \dots, m$

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq a_m$

(3) Non-negativity condition

$x_i \geq 0$ with $i = 1, \dots, n$

If a simplex tableau exists in canonical form and all coefficients of the non-basic variable ≥ 0 (≤ 0) are within the target function, then z_0 corresponds to the minimum (maximum) value of the function.

Example:

A firm manufactures two goods of the quantities x_1 and x_2 [measured in quantity units (QU)] using three machines, A, B and C, which are limited in their capacities [time units/month (TU/month)]. The maximum available capacities [measured in TU/month] as well as the production coefficients for the goods quantities [measured in TU/1 QU] are listed in the following table:

machine	**production coefficients [TU/QU]**		**max. capacities [TU/month]**
A	2 TU / 1 QU x_1	1 TU / 1 QU x_2	200
B	1 TU / 1 QU x_1	1 TU / 1 QU x_2	120
C	1 TU / 1 QU x_1	3 TU / 1 QU x_2	240
profit per unit	\$2 / 1 QU x_1	\$3 / 1 QU x_2	

The total profit in \$ has to be maximised.

Target function: $P(x_1,x_2) = 2x_1 + 3x_2 \Rightarrow$ max!

Constraints: $2x_1 + x_2 \leq 200$ with $x_1, x_2 \geq 0$

$x_1 + x_2 \leq 200$

$x_1 + 3x_2 \leq 120$

To set up the *simplex tableau*, *auxiliary variables* are introduced for the unused machine capacities (y_1, y_2, y_3) and the target function is set to zero. The subsequent equations are as follows:

$$\begin{array}{lll}
\text{(I)} & \Rightarrow & 2x_1 + x_2 + y_1 = 200 \\
\text{(II)} & \Rightarrow & x_1 + x_2 + y_2 = 120 \\
\text{(III)} & \Rightarrow & x_1 + 3x_2 + y_3 = 240 \\
\text{(IV)} & \Rightarrow & -2x_1 - 3x_2 + P = 0
\end{array}$$

From these equations, more precisely from the coefficients of the variables, the *simplex tableau* is created in the next step.

x_1	x_2	y_1	y_2	y_3	P	available capacities
2	1	1	0	0	0	200
1	1	0	1	0	0	120
1	3	0	0	1	0	240
−2	−3	0	0	0	1	0

non-basic variables *basic variables*

Since the first two columns in the 4th row contain negative values, the simplex tableau or the “solution” shown here is not optimal. For the first improvement step, the second column is selected as *pivot column*, because -3 represents the smallest value of all negative values within the bottom row. To select the *pivot row*, the value of the pivot column for each row is divided by the machine capacity. This identifies the (current) bottleneck capacities of the machines. The following is obtained:

Machine A: $$\frac{200\ \frac{\text{h}}{\text{month}}}{1\ \frac{\text{h}}{\text{QU}}} = 200\ \frac{\text{QU}}{\text{month}}$$

Machine B: $$\frac{120\ \frac{\text{h}}{\text{month}}}{1\ \frac{\text{h}}{\text{QU}}} = 120\ \frac{\text{QU}}{\text{month}}$$

Machine C: $$\frac{240\ \frac{\text{h}}{\text{month}}}{3\ \frac{\text{h}}{\text{QU}}} = 80\ \frac{\text{QU}}{\text{month}}$$

⇒ Machine C is the first to reach its capacity limit, so the 3rd row is our pivot row. To get a 1 at the position of the 3rd row (pivot row) and 2nd column (pivot column), the third row is multiplied by the corresponding inverse value (here $\frac{1}{3}$).

It follows:

x_1	x_2	y_1	y_2	y_3	P	available capacities
2	1	1	0	0	0	200
1	1	0	1	0	0	120
$1/3$	1	0	0	$1/3$	0	80
-2	-3	0	0	0	1	0

The next step is to create the unit vector for x_2 so that the pivot element (3rd row/2nd column) equals one (and all other column elements equal zero). This is achieved by adding the (-1)fold of the 3rd row to rows 1 and 2 and adding the 3fold of the 3rd row to the last row. The result is:

x_1	x_2	y_1	y_2	y_3	P	available capacities
$5/3$	0	1	0	$-1/3$	0	120
$2/3$	0	0	1	$-1/3$	0	40
$1/3$	1	0	0	$1/3$	0	80
-1	0	0	0	1	1	240

basic variables

There is still a negative value in row 4/column 1, therefore this simplex tableau or the solution shown here is not optimal. For the second improvement step, the 1st column is the *pivot column*, because -1 represents the last negative value within the bottom row. To select the *pivot row*, the value of the pivot column is divided by the machine capacity for each row again. The result is:

Machine A:

$$\frac{120\ \frac{\mathrm{h}}{\mathrm{month}}}{\frac{5}{3}\ \frac{\mathrm{h}}{\mathrm{QU}}} = 72\ \frac{\mathrm{QU}}{\mathrm{month}}$$

Machine B:

$$\frac{40\ \frac{\mathrm{h}}{\mathrm{month}}}{\frac{2}{3}\ \frac{\mathrm{h}}{\mathrm{QU}}} = 60\ \frac{\mathrm{QU}}{\mathrm{month}}$$

Machine C:

$$\frac{80\ \frac{\mathrm{h}}{\mathrm{month}}}{\frac{1}{3}\ \frac{\mathrm{h}}{\mathrm{QU}}} = 240\ \frac{\mathrm{QU}}{\mathrm{month}}$$

⇒ Machine B is the first to reach its capacity limit, so the 2nd row is our current pivot row. To get a 1 at the position of the 2nd row (pivot row) and 1st column (pivot column), the second column must be multiplied by the corresponding inverse value (here $\frac{3}{2}$).

It follows:

x_1	x_2	y_1	y_2	y_3	P	available capacities
$5/3$	0	1	0	$-1/3$	0	120
1	0	0	$3/2$	$-1/2$	0	60
$1/3$	1	0	0	$1/3$	0	80
-1	0	0	0	1	1	240

To create the unit vector for x_1 so that the pivot element 1st row/2nd column equals one (and all other column elements equal zero):

- add $(-\frac{5}{3})$fold of the 2nd row to the 1st row,
- add $(-\frac{1}{3})$fold of the 1st row to the 3rd row,
- add the 2nd row to the last row.

It follows:

x_1	x_2	y_1	y_2	y_3	P	available capacities
0	0	1	−2.5	0.5	0	20
1	0	0	1.5	−0.5	0	60
0	1	0	−0.5	0.5	0	60
0	0	0	1.5	0.5	1	300

basic variables

There are no more negative elements in the 4th row of the matrix. Now the optimal solution is reached. To optimise this problem discussed here, 60 units of x_1 and 60 units of x_2 have to be produced, whereby a capacity of 20 hours remains unused with machine A. The maximum profit is $\$300$.

8.2.6 Dual Simplex Algorithm

The *dual simplex algorithm* is used if negative values exist in the far right column of the simplex tableau. The aim is to replace all negative values in the right column by iteration with positive values so that in a first step a feasible solution is found. Afterwards, the primal (= original) simplex method can be used to find the optimal solution.

The dual simplex algorithm starts with the definition of a minimisation problem:[2]

---

[2] If the values in the right column of the simplex tableau are negative, then the quantities shown there are missing, i.e. they would have to be procured. Therefore, it inevitably follows that this is a minimisation problem.

Minimise $z = \sum_{j=1}^{n} c_j \cdot x_j$

by considering the constraints $\sum_{j=1}^{n} d_{ij} \cdot x_j \geq b_j$

$x_j \geq 0 \qquad c_j \geq 0$

(1) Target function

$z = z(x_i) = \sum_{i=1}^{n} c_i \cdot x_i \qquad \Rightarrow$ min!

$c_i \in \mathbb{R} =$ constant with $i = 1, \ldots, n$

(2) Constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m$$

$x_i \geq 0$ and $a_{ij} \in \mathbb{R} =$ constant with $j = 1, \ldots, m$

(3) Non-negativity condition

$x_i \geq 0$ with $i = 1, \ldots, n$

After transforming into a maximisation problem and the introduction of slack variables $y_1, ..., y_m$ the initial simplex tableau is generally represented as follows:

$x_1 \cdots x_n$	$y_1 \cdots y_m$	z	b
$a_{11} \cdots a_{1n}$	$1 \cdots 0$	0	$-b_1$
$\vdots \quad \vdots$	$\vdots \; \ddots \; \vdots$	$\vdots$	$\vdots$
$a_{m1} \cdots a_{mn}$	$0 \cdots 1$	0	$-b_m$
$c_1 \cdots c_n$	$0 \cdots 0$	1	z-value

Example: [3]

A firm produces three goods in quantities x_1, x_2, and x_3 [QU] using two machines, A and B, which are subject to minimum economic or technical utilization [ZE/day].

The production coefficients for the quantities of goods x_1, x_2, and x_3, measured in [TU/ 1 QU], are summarized in the table below:

machine	**produktion coefficients** [**TU/QU**]			**min. utilization** [**TU/day**]
A	4 TU / 1 QU x_1	2 TU / 1 QU x_2	5 TU / 1 QU x_3	12
B	2 TU / 1 QU x_1	3 TU / 1 QU x_2	1 TU / 1 QU x_3	8
unit costs in \$100	\$ 0.8 / 1 QU x_1	\$ 1.0 / 1 QU x_2	\$ 0.75 / 1 QU x_3	

The total costs in \$ are to be minimised.

Target function: $K(x_1, x_2, x_3) = 0.8x_1 + x_2 + 0.75x_3 \;\Rightarrow\;$ min!

Constraints:

$$4x_1 + 2x_2 + 5x_3 \geq 12$$

$$2x_1 + 3x_2 + x_3 \geq 8$$

with $x_i \geq 0 \qquad i = 1, 2, 3$

---

[3] The following example is based on Zimmermann, H.-J. (2005), p. 102 ff.

First the $\geq$ constraints are to be transformed into $\leq$ constraints by multiplication with -1. Furthermore, the minimisation problem has to be transformed into a maximisation problem. The non-negativity conditions remain. Now the standard form, as known from the *Linear Programming Approach*, exists:

$$-4x_1 - 2x_2 - 5x_3 \leq -12$$

$$-2x_1 - 3x_2 - x_3 \leq -8$$

$$-0.8x_1 - x_2 - 0.75x_3 = -K \Rightarrow \text{max!}$$

with $x_i \geq 0 \qquad i = 1, 2, 3$

Slack variables y_j are to be integrated so that equations can be achieved:

$$-4x_1 - 2x_2 - 5x_3 + y_1 = -12$$

$$-2x_1 - 3x_2 - x_3 + y_2 = -8$$

$$-0.8x_1 - x_2 - 0.75x_3 = -K = k \Rightarrow \text{max!}$$

with $x_i \geq 0 \qquad i = 1, 2, 3$

and $y_j \geq 0 \qquad j = 1, 2$

That equation system can be transformed into the following simplex tableau:

	x_1	x_2	x_3	y_1	y_2	b_i
I	-4	-2	-5	1	0	-12
II	-2	-3	-1	0	1	-8
III	0.8	1	0.75	0	0	0

The upper simplex tableau shows the non-basic variables x_1, x_2 and x_3 and the basic variables y_1 and y_2. The only possible solution on a mathematical point of view could be $x_1, x_2, x_3 = 0$ and $y_1 = -12, y_2 = -8$. It is dual, but not primal feasible. Since there are negative values at the right column the primal simplex algorithm cannot be used. The negative values have to be eliminated first.

In contrast to the *primal simplex algorithm*, first the pivot row is selected instead of the pivot column. For this, the row with the smallest negative value in the b_i column must be identified. If there are several rows with the same smallest value, one of them can be chosen.

In the table below, this row is marked in grey. -12 is the smallest negative value. In consequence the first row is the pivot row.

	x_1	x_2	x_3	y_1	y_2	b_i
I	-4	-2	-5	1	0	-12
II	-2	-3	-1	0	1	-8
III	0.8	1	0.75	0	0	0

Now each value from the target function row is to divide by the respective value in the pivot row, as illustrated at the following table:

	target function	row I	quotient
x_1	0.8	-4	-0.2
x_2	1	-2	-0.5
x_3	0.75	-5	-0.15

The column with the maximu negative value is determined as the pivot column. The pivot element is located where the pivot row crosses the pivot column. The pivot element is marked in dark grey at the table below.

	x_1	x_2	x_3	y_1	y_2	b_i
I	-4	-2	-5	1	0	-12
II	-2	-3	-1	0	1	-8
III	0.8	1	0.75	0	0	0

To get a 1 at the position of the 1st row (pivot row) and the 3rd column (pivot column), the 1st row is multiplied with the corresponding inverse value ($-\frac{1}{5}$).

It follows:

	x_1	x_2	x_3	y_1	y_2	b_i
I	$0.8 = -4 \cdot (-\frac{1}{5})$	0.4	1	-0.2	0	2.4
II	-2	-3	-1	0	1	-8
III	0.8	1	0.75	0	0	0

The next step is to create the unit vector for x_3. This is achieved by adding the 1st row to the 2nd row and adding the (-0.75)fold of the 1st row to the 3rd row. The result is:

	x_1	x_2	x_3	y_1	y_2	b_i
I	0.8	0.4	1	-0.2	0	2.4
II	-1.2	-2.6	0	-0.2	1	-5.6
III	$0.8 \cdot (-0.75) + 0.8 = 0.2$	$0.4 \cdot (-0.75) + 1 = 0.7$	$1 \cdot (-0.75) + 0.75 = 0$	$-0.2 \cdot (-0.75) + 0 = 0.15$	$0 \cdot (-0.75) + 0 = 0$	$2.4 \cdot (-0.75) + 0 = -1.8$

The right column still contains a negative value within the top two rows.

	x_1	x_2	x_3	y_1	y_2	b_i
I	0.8	0.4	1	−0.2	0	2.4
II	−1.2	−2.6	0	−0.2	1	−5.6
III	0.2	0.7	0	0.15	0	−1.8

In consequence the current status cannot be the optimal solution.

The following iteration starts with the identification of the next pivot element. -5.6 is now the smallest negative value of the first two rows. The second row is the pivot row.

	target function	row II	quotient
x_1	0.2	−1.2	−0.167
x_2	0.7	−2.6	−0.269
x_3	0	0	0

The column with the maximum negative value (= the minimum positive absolute value) is determined as the pivot column. The pivot element is located where the pivot row crosses the pivot column. The pivot element is marked in dark grey at the table below.

	x_1	x_2	x_3	y_1	y_2	b_i
I	0.8	0.4	1	−0.2	0	2.4
II	−1.2	−2.6	0	−0.2	1	−5.6
III	0.2	0.7	0	0.15	0	−1.8

To get a 1 at the position of the 2nd row (pivot row) and the 1st column (pivot column), the 2nd row is multiplied with the corresponding inverse value ($-\frac{1}{1.2}$).

It follows:

	x_1	x_2	x_3	y_1	y_2	b_i
I	0.8	0.4	1	-0.2	0	2.4
II	1	2.1667	0	0.1667	-0.8333	4.6667
III	0.2	0.7	0	0.15	0	-1.8

The next step is to create the unit vector for x_1. This is achieved by adding the (-0.8)fold of the 2nd row to the 1st row and adding the (-0.2)fold of the 2nd row to the 3rd row. The result is:

	x_1	x_2	x_3	y_1	y_2	b_i
I	0	-1.3333	1	-0.3333	0.6666	-1.3333
II	1	2.1667	0	0.1667	-0.8333	4.6667
III	0	0.2667	0	0.1167	0.1667	-2.7333

The right column still contains a negative value within the top two rows. In consequence the current status cannot be the optimal solution.

The following iteration starts with the identification of the next pivot element. $-1,333$ is now the smallest negative value at the first two rows. The first row is the pivot row.

	target function	row I	quotient
x_1	0	0	0
x_2	0.2667	-1.3333	-0.2
x_3	0	1	0

The second column is determined as the pivot column. The pivot element is located where the pivot row crosses the pivot column. The pivot element is marked in dark grey at the table below.

	x_1	x_2	x_3	y_1	y_2	b_i
I	0	-1.3333	1	-0.3333	0.6666	-1.3333
II	1	2.1667	0	0.1667	-0.8333	4.6667
III	0	0.2667	0	0.1167	0.1667	-2.7333

To get a 1 at the position of the 1st row (pivot row) and the 2nd column (pivot column), the 1st row is multiplied with the corresponding inverse value ($-\frac{1}{1.3333}$).

It follows:

	x_1	x_2	x_3	y_1	y_2	b_i
I	0	1	-0.75	0.25	-0.5	1
II	1	2.1667	0	0.1667	-0.8333	4.6667
III	0	0.2667	0	0.1167	0.1667	-2.7333

The next step is to create the unit vector for x_2. This is achieved by adding the (-2.1667)fold of the 1st row to the 2nd row and adding the (-0.2667)fold of the 1st row to the 3rd row. The result is:

	x_1	x_2	x_3	y_1	y_2	b_i
I	0	1	-0.75	0.25	-0.5	1
II	1	0	1.625	-0.375	0.25	2.5
III	0	0	0.2	0.05	0.3	-3

The right column contains no more negative values at the first to rows. So the optimal solution has been identified.

$$x_1 = 2.5$$

$$x_2 = 1$$

$$k = -K = -3 \quad \Rightarrow \quad K = 3 \cdot \$100 = \$300$$

To minimise the total costs, 2.5 QU of x_1, 1 QU of x_2 and no QU of x_3 are to be produced per day. The resulting minimum costs are $\$300$ per day.

Chapter 9

Functions

9.1 Introduction

A function $f = f(x)$ is an unique assignment of “x to f of x”: $x \mapsto f(x)$. A function $y = f(x)(x \mapsto y)$ has exactly one dependent value y assigned to each independent value, also called “argument” x.

Set Notation

$$f = \{(x,y) \mid y = f(x),\ x \in X,\ y \in Y\}$$

with $X =$ set of all x-values (domain)

$Y =$ set of all y-values (codomain)

Graphical Representation

In the graphical representation of the function $y = f(x)$ in the rectangular coordinate system, each pair of values (x,y) is uniquely assigned to a point $P(x,y)$ on the x-y-plane. This results in the so-called *function curve* or *function graph*.

Domain D_f

D_f – also $D(f)$ – contains the set of all (allowed or desired) x-values. In the Cartesian coordinate system, D_f describes the possible x-coordinates of f.

F. W. Peren, *Math for Business and Economics*,
https://doi.org/10.1007/978-3-662-68860-1_9

Codomain C_f

C_f – also $C(f)$ – contains the set of all function values (y-values). Graphically, C_f describes the y-coordinates generated by the function f.

Function Value

$f(x_0)$ is the function value of the function $f = f(x)$ located at $x = x_0$.

Function Graph

Graphical representation of the function f in the Cartesian coordinate system (function curve/function graph)

$y = ax + b$ **linear function**

a corresponds to the (constant) slope of the function,
b corresponds to the y-intercept (point of intersection with the y-axis) of the function.

Examples:

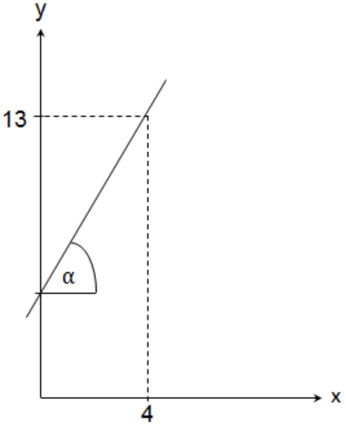
- $y = 2x + 5$ **linear function**

  Slope: tan $\alpha = 2$

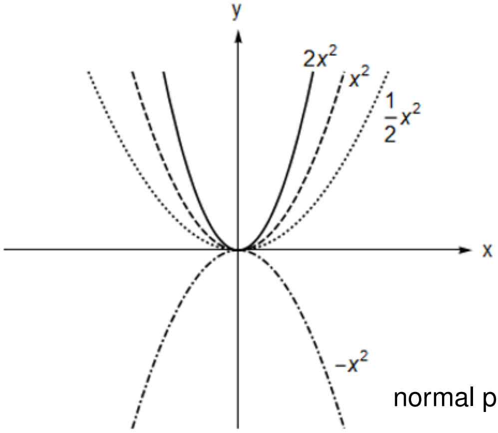
  y-intercept: $y(0) = 5$

  Function value at position $x_0 = 4$

  $y_0 = y(4) = 2 \cdot 4 + 5 = 13$



- $y = x^2$ **quadratic function**



(stretched parabola)

(normal parabola)

(compressed parabola)

normal parabola opened downwards

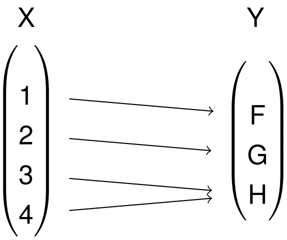
Equality of Functions

The following two functions $f = f(x)$ and $g = g(x)$ are equal if $f(x) = g(x)$ for all $x \in D$ with $D = D_f = D_g$.

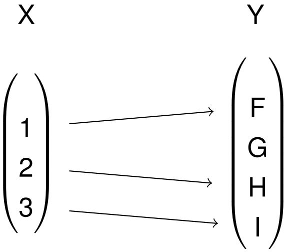
Surjection, Injection, Bijection

A function $f = f(x)$ with $D_f = X$ and $C_f = Y$ ($X =$ set of all x-values; $Y =$ set of all y-values) is

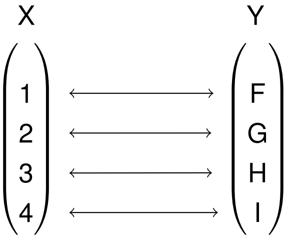
surjective, if for every $y \in Y$ there is at least one $x \in X$ that applies: $y = f(x)$. A surjective function is a *unique* illustration.



injective, if for each $y \in Y$ there is at most one $x \in X$ (i.e. possibly none at all) that applies: $y = f(x)$. An injective function is also a *unique* illustration.



bijective, if the function $f = f(x)$ is *injective* and *surjective*. The function is *reversible*, i.e. *one-to-one*. This means that each $x \in X$ is assigned to exactly one $y \in Y$ and each $y \in Y$ is assigned to exactly one $x \in X$. The direction of causality is reversible, i.e. both variables can become independent values (arguments). $x = x(y)$ or $y = y(x)$ applies. Both representations are possible and thus each, by definition, forms a function, i.e. each is an unique representation. It is therefore referred to as a one-to-one function.



9.1.1 Composition of Functions

Given are the functions

$f = f(x)$ $\qquad x \mapsto f(x)$ $\qquad$ with $x \in D_f$ and $f(x) \in C_f$

and

$g = g(x)$ $\qquad x \mapsto g(x)$ $\qquad$ with $x \in D_g$ and $g(x) \in C_g$

Compositions

- $f(g(x)) = f \circ g = fg$
  read as “f of g” or “f about g”; $x \mapsto f(g(x))$

  g as the inner function, f as the outer function, for the composition $f(g(x))$ applies:

  $D(f \circ g) = D_g$ domain

  $W(f \circ g) = \{f(g(x)) \mid x \in D_g\} \subseteq D_f$
  Codomain $=$ proper or improper subset of the domain of the function f

- $g(f(x)) = g \circ f = gf$
  read as “g of f” or “g about f”; $x \mapsto g(f(x))$

  f as the inner function, g as the outer function, for the composition $g(f(x))$ applies:

  $D(g \circ f) = D_f$ domain

  $W(g \circ f) = \{g(f(x)) \mid x \in D_f\} \subseteq D_g$
  Codomain $=$ proper or improper subset of the domain of the function g

Note:

To compose two functions, the codomain of the inner function must be (proper or improper) subset of the domain of the outer function.

Examples:

(1)

$$f(x) = \boxed{\boxed{x}^{4} + 1}$$

$$g(x) = \boxed{\sqrt{\boxed{x}}}$$

$$f(g(x)) = \boxed{\left(\boxed{\sqrt{x}}\right)^{4} + 1}$$

$$g(f(x)) = \boxed{\sqrt{\boxed{x^{4} + 1}}}$$

(2)

$$f(x) = \boxed{\boxed{x}^{100}}$$

$$g(x) = \boxed{\boxed{x}^{2} + e}$$

$$f(g(x)) = \boxed{\left(\boxed{x^{2} + e}\right)^{100}}$$

$$g(f(x)) = \boxed{\left(\left(\boxed{x^{100}}\right)^{2} + e\right)}$$

General Composition of (n+1)-Functions

$$f = f(g_1(g_2(g_3 \dots g_n(x))))$$

Example:

$$f(g_1(g_2(x))) = \boxed{\left[\boxed{\ln\left(\boxed{x^{4} + 1}\right)}\right]^{8}}$$

with	$f(x) = x^8$	(power function)
	$g_1(x) = lnx$	(logarithmic function)
	$g_2(x) = x^4 + 1$	(polynomial function/ polynomial of 4th degree)

$$f(g_1(x)) = (lnx)^8 \qquad \Rightarrow \qquad f'(g_1(x)) = 8\cdot(lnx)^7\cdot\frac{1}{x}$$

$$g_1(g_2(x)) = ln(x^4+1) \qquad \Rightarrow \qquad g_1'(g_2(x)) = \frac{1}{x^4+1}\cdot 4x^3$$

$$g_2(x) = x^4+1 \qquad \Rightarrow \qquad g_2'(x) = 4x^3$$

$$g_2\left(g_1(f(x))\right) = \left(ln(x^8)\right)^4+1 \qquad \Rightarrow \qquad g_2'\left(g_1(f(x))\right) = 4\left(ln(x^8)\right)^3\cdot\frac{1}{x^8}\cdot 8x^7$$

9.1.2 Inverse Function

The independent and dependent values are reversed if the inverse function is formed:

$$f\colon y(x) \quad \Rightarrow \qquad f^{-1}: x(y)$$

The prerequisite is that the function f is one-to-one, i.e. that the unique function is again unique after its reversal. Every strictly monotonic function is reversible.

Formation of the Inverse Function

$y = f(x)$ solving for x: $\quad x = f^{-1}(y)$

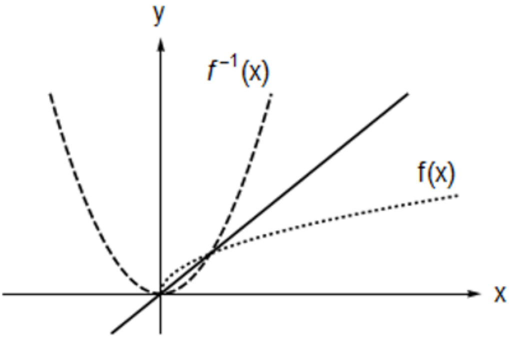
$D_{f^{-1}} = C_f \qquad C_{f^{-1}} = D_f$

Graphically, the inversion of a function is a reflection of the original function at the 45° angle bisector in an equally dimensioned Cartesian coordinate system.

Example:

$f(x): y = \sqrt{x}$

$D_f = \mathbb{R}_0^+ \quad C_f = \mathbb{R}$



Solving for x: $x = y^2$ and swapping the variable name:

$f^{-1}(x): y = x^2$

$D_{f^{-1}} = C_f = \mathbb{R}$

$C_{f^{-1}} = D_f = \mathbb{R}_0^+$

9.2 Classification of Functions

(real) functions

rational functions	non-rational functions
a) polynomial functions	a) power functions/ root function
b) broken rational functions	b) transcendental functions
• proper broken functions	• exponential functions
• improper broken functions	• logarithmic functions
	• trigonometric functions
	• hyperbolic functions/ area functions

Real Functions

The domain D_f of real functions, $f = f(x)$, includes the set of real numbers or a subset thereof: $D_f \subseteq \mathbb{R}$.

9.2.1 Rational Functions

polynomial functions

broken rational functions

9.2.1.1 Polynomial Functions

= n^{th}- degree polynomials

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = \sum_{k=0}^{n} a_k x^k$$

with $n \in \mathbb{N};\ a_i \in \mathbb{R};\ a_n \neq 0$

9.2.1.2 Broken Rational Functions

= a fraction of polynomial functions

= a fraction of polynomials

$$y = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0} = \frac{\sum_{k=0}^{n} a_k x^k}{\sum_{l=0}^{m} b_l x^l}$$

with $n, m \in \mathbb{N};\ a_i, b_j \in \mathbb{R};\ a_n, b_m \neq 0$

Proper Broken Rational Functions

= proper broken rational functions with $n < m$
(degree of the numerator < degree of the denominator)

Example: $f(x) = \dfrac{x^1 + 6}{x^2 + x}$

Improper Broken Rational Functions

= improper broken rational functions with $n > m$
(degree of the numerator $>$ degree of the denominator)

Example: $f(x) = \dfrac{x^2+x}{x^1+6}$

Characteristics

- constraints in the domain
- discontinuities
- poles
- asymptotes

Constraints in the domain

All x-values, for which the denominator is equal to 0, must be excluded from the domain of the function.

$\mathbb{D} = \mathbb{R} \backslash \{$zeros of the denominator$\}$

Example: $f(x) = \dfrac{(x-5)}{(x+1)^2}$

Set the denominator to 0:

$$\begin{aligned} && (x+1)^2 &= 0 && \mid \sqrt{()} \\ \Leftrightarrow && x+1 &= 0 && \mid -1 \\ \Leftrightarrow && x &= -1 && \end{aligned}$$

$$f(-1) = \frac{-1-5}{(-1+1)^2} = \frac{-6}{0^2} \qquad \rightarrow \mathbb{D} = \mathbb{R} \backslash \{-1\}$$

Discontinuities

Discontinuities in broken rational functions usually refer to *poles*.

Example: $f(x) = \dfrac{x^3}{x-1}$ with $D_f = \mathbb{R}\backslash\{1\}$

$\Rightarrow$ Pole at $x = 1$

Exceptions are the so-called *removable discontinuities*.

A discontinuity is removable if it is possible to simplify the corresponding function term. This means that the original function also has a zero at this position, which is not defined.

Example:

$$f(x) = \frac{1-x^2}{x^2-x-2}$$

$$\Leftrightarrow \qquad \frac{(1+x)\cdot(1-x)}{(x-2)\cdot(x+1)} = \frac{(1-x)}{(x-2)}$$

$\Rightarrow$ zeros at $x = -1$ and $x = 1$
discontinuities at $x = -1$ and $x = 2$

$\Rightarrow$ $D_f = \mathbb{R}\backslash\{-1;2\}$

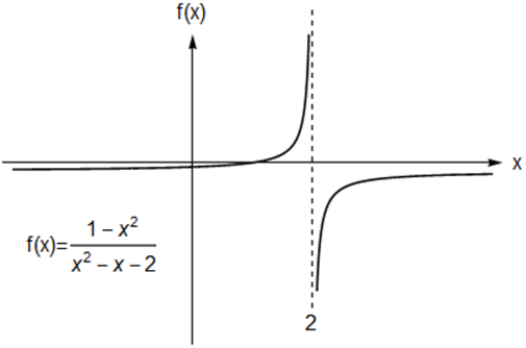
The discontinuity at $x = -1$ is removable; i.e. the term $(x+1)$ can be simplified. However, the initial function $f(x)$ has a remaining discontinuity at $x = -1$, which has to be excluded from the domain.

The discontinuity at $x =$2 is not removable; i.e. the term cannot be simplified. There is a pole here.

The type of pole can be determined by the left and right limits (limes) of the function:

$$\lim_{x \to 2^-} \approx f(1.9999) = +\infty$$

$$\lim_{x \to 2^+} \approx f(2.0001) = -\infty$$

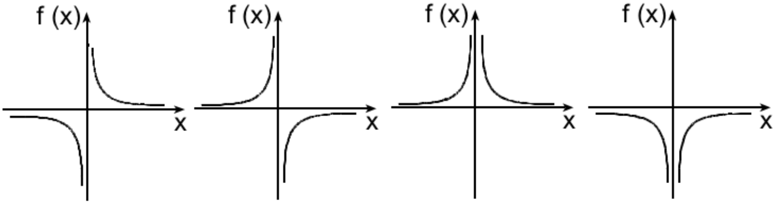


There are four types of poles:

Example:

$$f(x) = \frac{5}{x} = 5x^{-1} \text{ with } D_f = \mathbb{R}\backslash\{0\}$$

The pole is at $x = 0$. In such case, the function graph of $f(x)$ at $x = 0$ could develop as follows:



These four possibilities of a pole exist. In the given case for $f(x) = 5x^{-1}$, the graph displays the same behaviour shown in the first graph on the left.

9.2.2 Non-rational Functions

power functions/root functions/transcendental functions

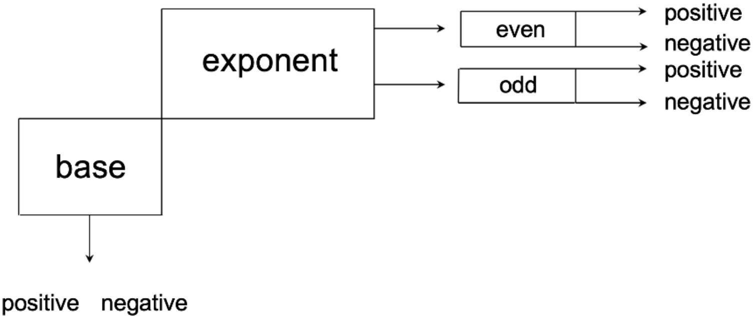
9.2.2.1 Power Functions

$f(x) = ax^k \quad \text{with} \quad k \in \mathbb{R}; \quad a \in \mathbb{R}; \quad D_f = \mathbb{R}$

The independent variable x of the function $f = f(x)$ forms the basis (of the exponential expression).

The Form of Power Functions

$f(x) = ax^k$

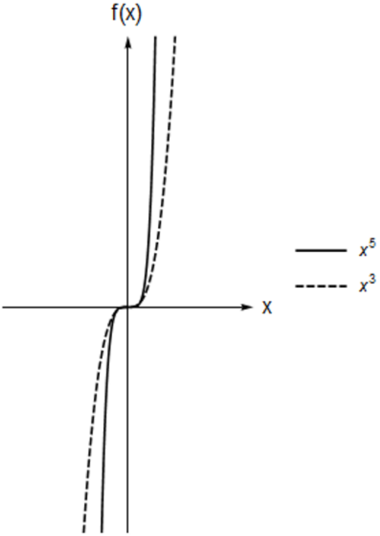


Example:

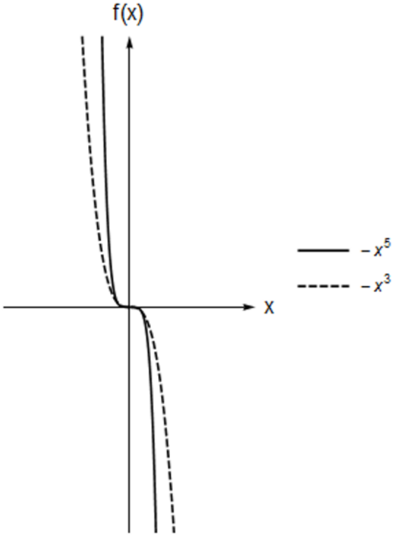
k is positive:

$f(x) = x^2$	normal parabola
$f(x) = -x^2$	normal parabola opened downwards
$\mid a \mid < 1$	compressed parabola
$\mid a \mid > 1$	stretched parabola

$f_1(x) = x^3; \quad f_2(x) = x^5$



$f_1(x) = -x^3; \quad f_2(x) = -x^5$



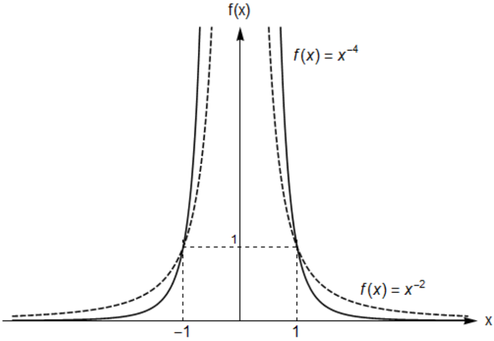
k is negative:

even power function

⇒ pole (hyperbola) without change of sign

$f(x) = x^{-2} = \frac{1}{x^2};$

$f(x) = x^{-4} = \frac{1}{x^4};$

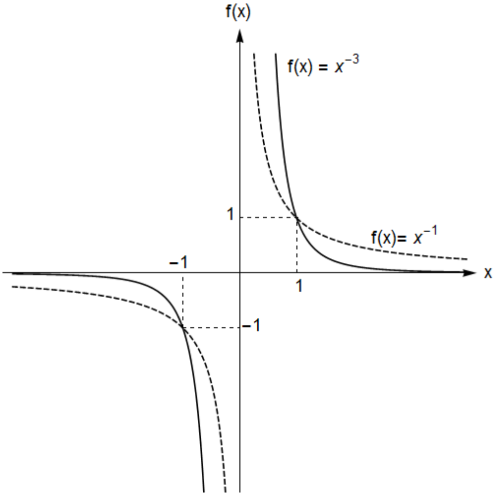


odd power function

⇒ pole (hyperbola) with change of sign

$f(x) = x^{-1} = \frac{1}{x^1};$

$f(x) = x^{-3} = \frac{1}{x^3};$



9.2.2.2 Root Function

$f(x) = a\sqrt[l]{x^k} = ax^{\frac{k}{l}}$ with $k \in \mathbb{R}$;
$l \in \mathbb{N}$ with $n \geq 1$;
$a \in \mathbb{R}$ with $a \neq 0$;
$D_f = \mathbb{R}_0^+$

The independent variable x is in the radicand. The root function is the inverse function of the corresponding power function.

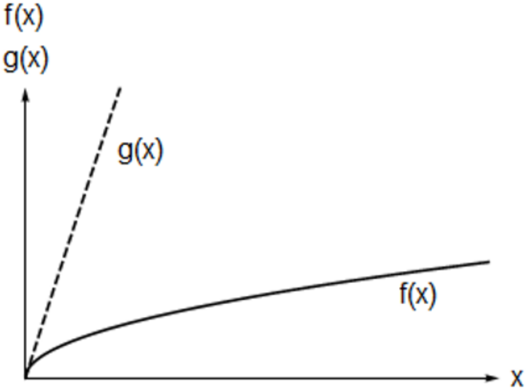
Examples:

$f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$f^{-1}(x) = y^2$

$g(x) = 8x^{\frac{5}{7}} = 8\sqrt[7]{x^5}$

$g^{-1}(x) = \sqrt[5]{\left(\frac{y}{8}\right)^7}$



9.2.2.3 Transcendental Functions

Functions that are not algebraic are *transcendental functions*.

9.2.2.3.1 Exponential Functions

$$f(x) = a^x \qquad \text{with} \qquad a \in \mathbb{R}^+, \qquad a \neq 1$$

$$D_f = \mathbb{R}, \qquad C_f = \mathbb{R}^+$$

The independent variable x is in the exponent.

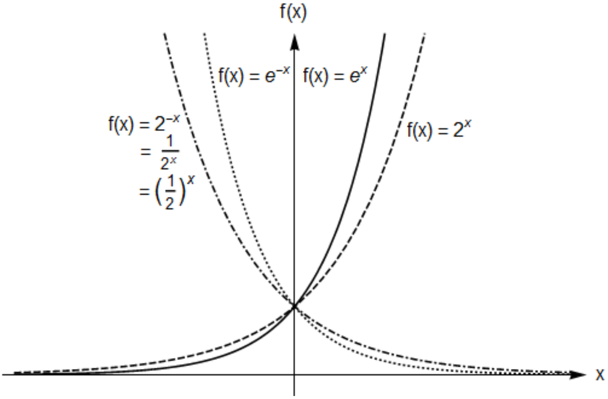
Characteristics:

$a > 1$	f strictly monotonically increasing
$0 < a < 1$	f strictly monotonically decreasing
$x = 0$	$f(0) = 1$

Since $a > 0$ and therefore $f(x) = a^x$ are always positive, the following applies:

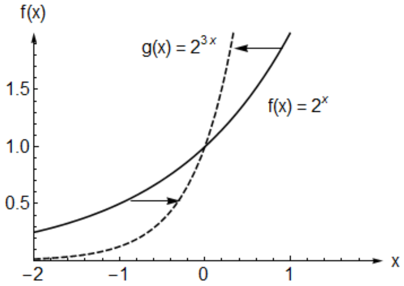
- The values of each exponential function are positive.
- The value set of each exponential function is the set $\mathbb{R}^+$.
- There are no zeros. The horizontal asymptote corresponds to the x-axis.
- The course of the exponential function is dependent on the base a.

Examples:



Compression/Stretch

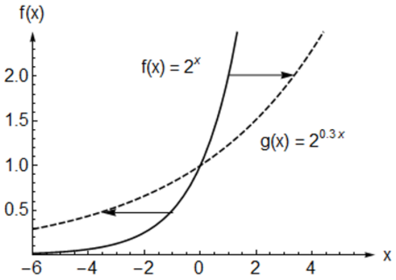
Compressed towards the x-axis



⇒ exponent is multiplied by a number n, with $n > 1$

$$f(x) = a^x \quad \rightarrow \quad g(x) = a^{n \cdot x}$$

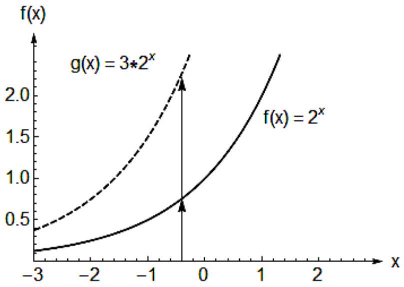
Stretched towards the x-axis



⇒ exponent is multiplied by a number n, with $0 < n < 1$

$$f(x) = a^x \quad \rightarrow \quad g(x) = a^{n \cdot x}$$

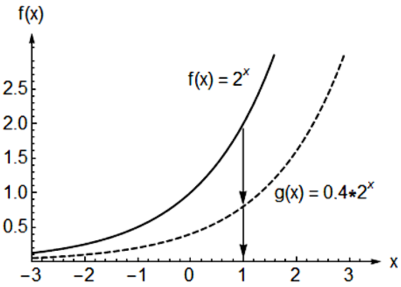
Stretched towards the y-axis



⇒ function $f(x)$ is multiplied by a number n, with $n > 1$

$$f(x) = a^x \quad \rightarrow \quad g(x) = n \cdot a^x$$

Compressed towards the y-axis

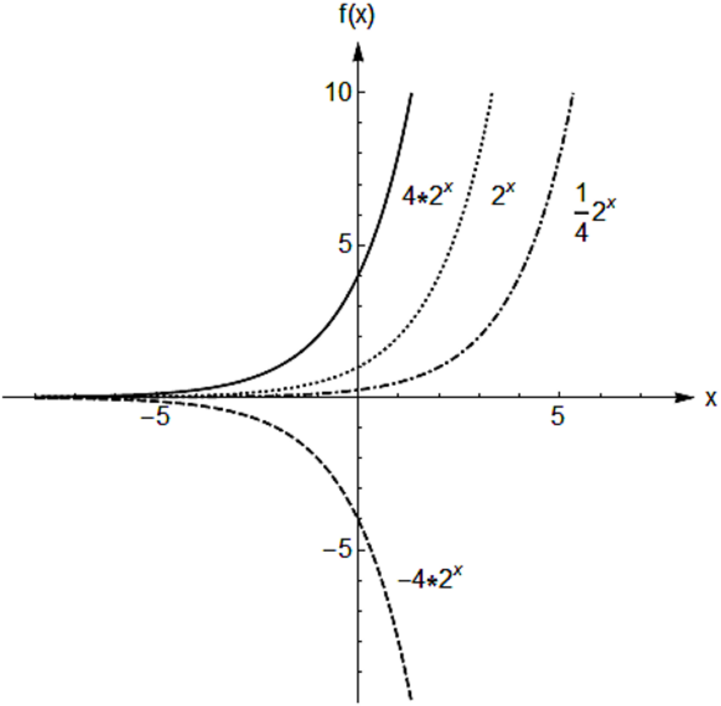


⇒ function $f(x)$ is multiplied by a number n, with $0 < n < 1$

$$f(x) = a^x \quad \rightarrow \quad g(x) = n \cdot a^x$$

Reflections

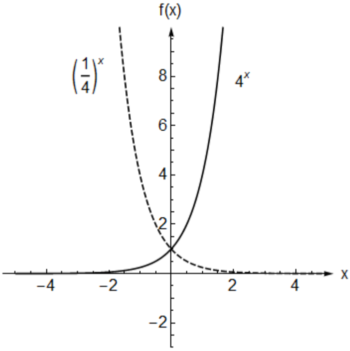
Reflection across the x-axis

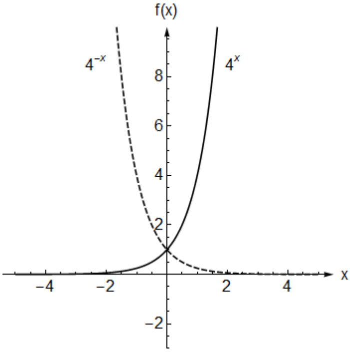


$f(x) = a^x$ multiplied by -1 equals: $g(x) = -a^x$

$y = 4 \cdot 2^x \quad \rightarrow \quad y = -4 \cdot 2^x$

Reflection across the y-axis





1st possibility

Form the inverse of the base

$$f_1(x) = a^x$$

$$f_2(x) = \left(\frac{1}{a}\right)^x$$

2nd possibility

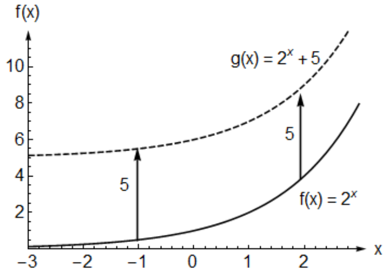
Multiply the exponent by -1

$$g_1(x) = a^x$$

$$g_2(x) = a^{-x}$$

Shift

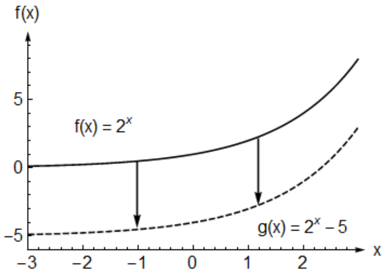
Upward shift



⇒ add a constant c to the function, with $c > 0$

$$f(x) = a^x \quad \rightarrow \quad g(x) = a^x + c$$

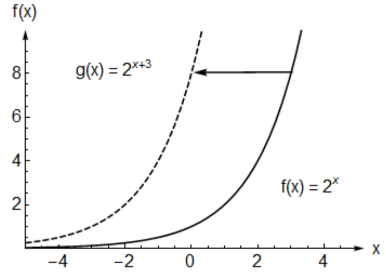
Downward shift



⇒ add a constant c to the function, with $c < 0$

$$f(x) = a^x \quad \rightarrow \quad g(x) = a^x - c$$

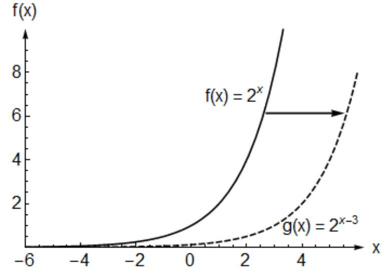
Left shift



⇒ add a constant c to the exponent, with $c > 0$

$$f(x) = a^x \quad \rightarrow \quad g(x) = a^{x+c}$$

Right shift



⇒ add a constant c to the exponent, with $c < 0$

$$f(x) = a^x \quad \rightarrow \quad g(x) = a^{x-c}$$

Natural Exponential Function

$f(x) = e^x$ Euler's number

$\rightarrow e = 2.718281828459$

Application

- when modeling continuous growth
- when modeling continuous decay

Requirements

- within the same time intervals
- with a constant growth factor

9.2.2.3.2 Logarithmic Functions

$f(x) = log_a x$ with $a \in \mathbb{R}^+, \quad a \neq 1; \quad D_f = \mathbb{R}^+; \quad C_f = \mathbb{R}$

read as “logarithm of x to the base a”.

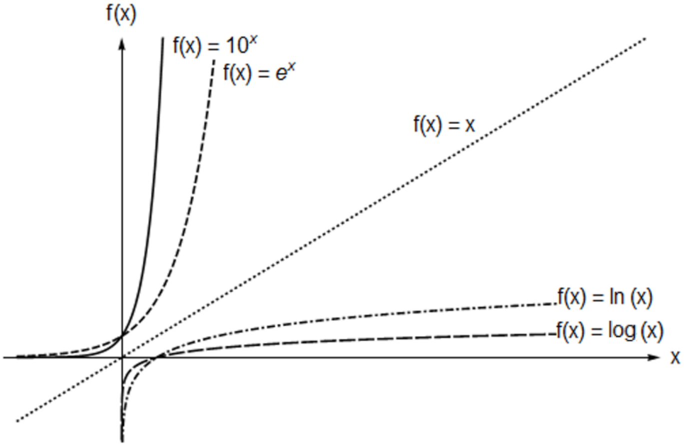
The logarithmic function is the inverse function of the corresponding exponential function.

Characteristics:

$a > 1$	f strictly monotonically increasing
$0 < a < 1$	f strictly monotonically decreasing
$x = 1$	$f(x) = 0$

The horizontal asymptote is the y-axis.

Examples:



Calculation of an arbitrary logarithm

$$log_{4711} 13 = \frac{log\ 13}{log\ 4711} = \frac{ln\ 13}{ln\ 4711}$$

Logarithmizing

$a^x = c$ → $a =$ base
$x =$ exponent
$c =$ power

$log_a(x) = c$ → $x =$ anti-logarithm
$c =$ logarithm

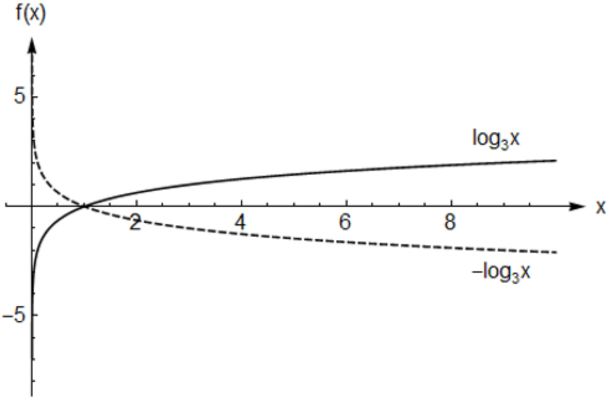
Example:

Searched:	$log_5 125 = x$
Equivalent equation:	$5^x = 125$
In words:	By which power x must 5 be raised to obtain the number 125?
Solution:	$x = 3 = \frac{\log 125}{\log 5} = \frac{\ln 125}{\ln 5}$

Reflection

Reflection across the x-axis

$f(x) = log_a c$ multiplied by -1, gives $g(x) = -log_a c$



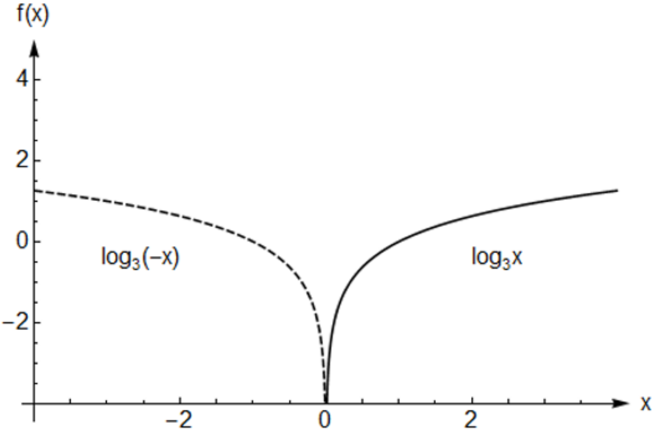
Reflection across the y-axis

$f(x) = log_a x$

The anti-logarithm (x) is given a negative sign

$g(x) = log_a(-x)$

Domain of $g(x)$: $D_f = R < 0$

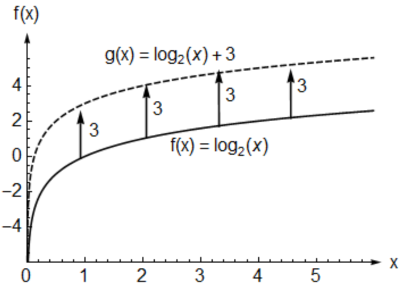


$g(-1) = log_3(-(-1))$

$g(-1) = log_3(1)$

$g(-1) = 0$

Shift

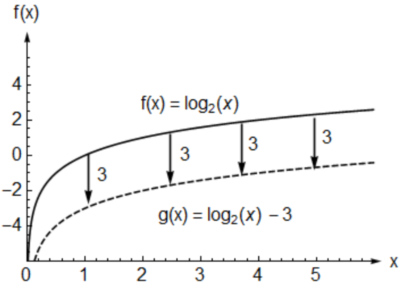


Upward shift

$f(x) = log_a$

⇒ add a constant c to the function, with $c > 0$

$$g(x) = log_a(x) + c$$

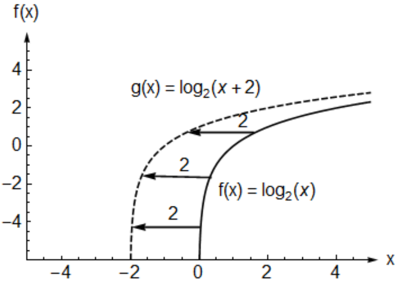


Downward shift

$f(x) = log_a$

⇒ substract a constant c from the function, with $c > 0$

$$g(x) = log_a(x) - c$$

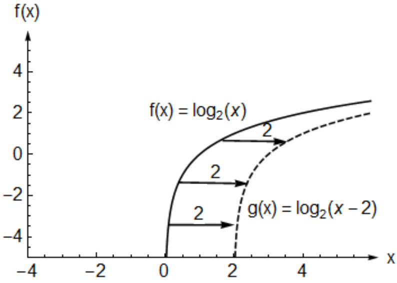


Left shift

$f(x) = log_a$

⇒ add a constant c to the anti-logarithm, with $c > 0$

$$g(x) = log_a(x + c)$$



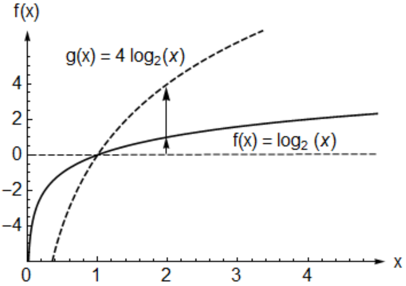
Right shift

$f(x) = log_a$

⇒ substract a constant c from the anti-logarithm, with $c > 0$

$$g(x) = log_a(x - c)$$

Stretch/Compression

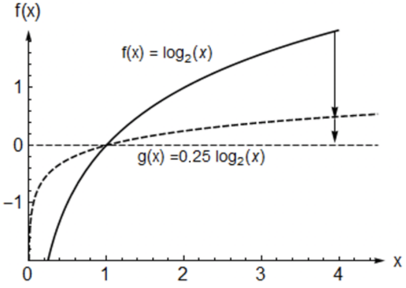


Stretch toward the y-axis

$$f(x) = log_a x$$

⇒ function is multiplied by a factor n, with $n > 1$

$$g(x) = n \cdot log_a x$$

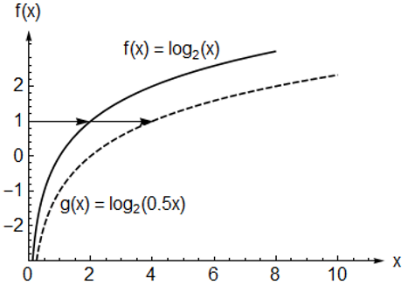


Compression toward the y-axis

$$f(x) = log_a x$$

⇒ function is multiplied by a factor n, with $0 < n < 1$

$$g(x) = n \cdot log_a x$$

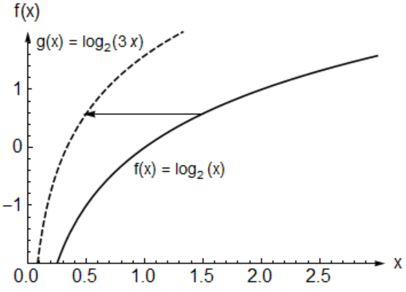


Stretch toward the x-axis

$$f(x) = log_a x$$

⇒ anti-logarithm (x) is multiplied by a factor n, with $0 < n < 1$

$$g(x) = log_a(n \cdot x)$$



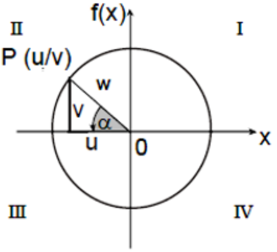
Compression toward the x-axis

$$f(x) = log_a x$$

⇒ anti-logarithm (x) is multiplied by a factor n, with $n > 1$

$$g(x) = log_a(n \cdot x)$$

9.2.2.4 Trigonometric Functions (Angle Functions/Circular Functions)



α is an arbitrary - counterclockwise - angle in a circle of which the center is located at the origin of a Cartesian coordinate system. α defines the angle formed by the (boundary) points A and P situated on the circle and the origin 0: $\triangleleft\, AOP$. The (end) point P has the coordinates (u/v), thus α can also be written as $\triangleleft\,(u/v)$.

Angle $\alpha = \triangleleft\,(u,v) = \triangleleft\, AOP$

The so-called special case *unit circle* occurs when the circle has a radius of one unit, $x = f(x) = 1$

In general, i.e. for any angle, the following applies:

- $f(x) = \sin\ \alpha = \frac{v}{w}$ $\quad D_f = \mathbb{R}$ $\quad C_f \in [-1,1]$
- $f(x) = \cos\ \alpha = \frac{u}{w}$ $\quad D_f = \mathbb{R}$ $\quad C_f \in [-1,1]$
- $f(x) = \tan\ \alpha = \frac{v}{u}$ $\quad D_f = \mathbb{R}\backslash\{x \,|\, x = \frac{\pi}{2} + k\pi\}$ $\quad C_f \in (-\infty,\infty) = \mathbb{R}$
- $f(x) = \cot\ \alpha = \frac{u}{v}$ $\quad D_f = \mathbb{R}\backslash\{x \,|\, x = k\pi\}$ $\quad C_f \in (-\infty,\infty) = \mathbb{R}$

$$\tan\alpha = \frac{\sin\ \alpha}{\cos\ \alpha} \qquad \cot\alpha = \frac{\cos\ \alpha}{\sin\ \alpha}$$

On the unit circle ($r = 1$) applies:

$$\sin\alpha = v \qquad (y\text{-coordinate of } P)$$

$$\cos\alpha = u \qquad (x\text{-coordinate of } P)$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \qquad \text{with} \quad \alpha \neq (2k+1)\cdot 90^\circ;\ k \in \mathbb{Z}$$

$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{\cos\alpha}{\sin\alpha} \qquad \text{with} \quad \alpha \neq k \cdot 180^\circ;\ k \in \mathbb{Z}$$

The following relationships also apply on the unit circle:

$$\cos^2\alpha + \sin^2\alpha = 1$$

$$\sin\alpha = \frac{\tan\alpha}{\pm\sqrt{1+\tan^2\alpha}}$$

$$\sin\alpha = \cos(90^\circ - \alpha)$$

$$\cos\alpha = \frac{1}{\pm\sqrt{1+\tan^2\alpha}}$$

$$\cos\alpha = \sin(90^\circ - \alpha)$$

$$\tan\alpha = \frac{\sin\alpha}{\pm\sqrt{1-\sin^2\alpha}}$$

$$\tan\alpha = \cot(90^\circ - \alpha)$$

The sign of the root depends on whether it is located in the positive (positive root) or negative (negative root) area of the coordinate system.

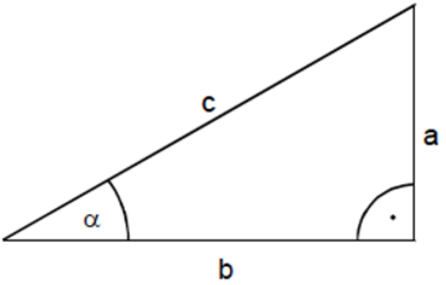
In a right-angled triangle applies: (of α)

$$\sin\alpha = \frac{a}{c} = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\alpha = \frac{b}{c} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\alpha = \frac{a}{b} = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cot\alpha = \frac{b}{a} = \frac{\text{adjacent}}{\text{opposite}}$$

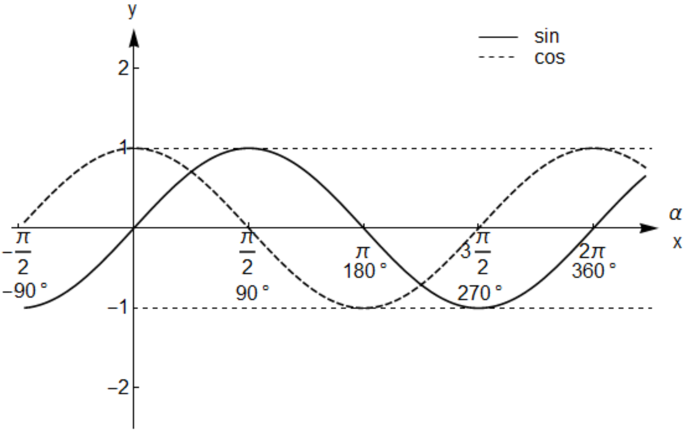


Representation of the Trigonometric Functions:

$f(\alpha) = \sin\alpha$

or

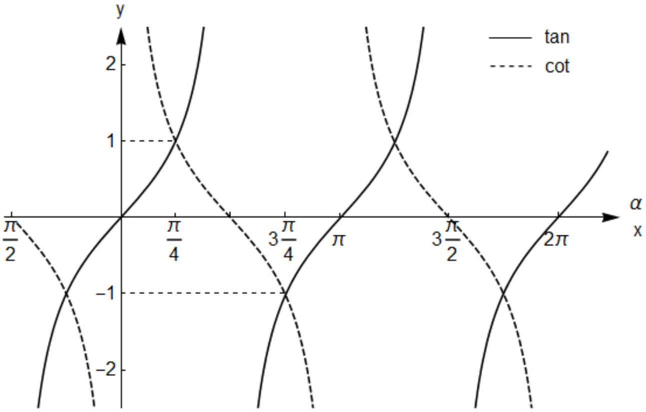
$f(\alpha) = \cos\alpha$



$f(\alpha) = \tan\alpha$

or

$f(\alpha) = \cot\alpha$



Characteristics of Trigonometric Functions $(k \in \mathbb{Z})$

	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
Domain	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}\backslash\{x \mid x = \frac{\pi}{2} + k\pi\}$	$\mathbb{R}\backslash\{x \mid x = k\pi\}$
Codomain	$[-1.1]$	$[-1.1]$	$(-\infty, \infty) = \mathbb{R}$	$(-\infty, \infty) = \mathbb{R}$
Zeros	$k\pi$	$\frac{\pi}{2} + k\pi$	$k\pi$	$\left(\frac{1}{2} + k\right) \cdot \pi$
Poles	-	-	$(2k+1) \cdot \left(\frac{\pi}{2}\right)$	$k\pi$
Extremes	*see below*	*see below*	-	-
Inflection points	$k \cdot \pi$ *see below*	$\frac{\pi}{2} + k \cdot \pi$ *see below*	$k \cdot \pi$ *see below*	$\frac{\pi}{2} + k \cdot \pi$ *see below*
Asymptotes	-	-	$(2k+1) \cdot \frac{\pi}{2}$ *see below*	$k \cdot \pi$ *see below*
Periods	2π	2π	π	π

Extremes

$\sin\alpha$: maxima: $\sin\left(\frac{4k+1}{2}\cdot\pi\right) = 1$ with $k \in \mathbb{Z}$

i.e. $\left\{\ldots,\ -\frac{7\pi}{2},\ -\frac{3\pi}{2},\ \frac{\pi}{2},\ \frac{5\pi}{2},\ \frac{9\pi}{2},\ \ldots\right\}$

are maxima of $\sin\alpha$

minima: $\sin\left(\frac{4k-1}{2}\cdot\pi\right) = -1$ with $k \in \mathbb{Z}$

i.e. $\left\{\ldots,\ -\frac{9\pi}{2},\ -\frac{5\pi}{2},\ -\frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{7\pi}{2},\ \ldots\right\}$

are minima of $\sin\alpha$

$\cos\alpha$: maxima: $\cos(2k\cdot\pi) = 1$ with $k \in \mathbb{Z}$

i.e. $\{\ldots,\ -4\pi, -2\pi,\ 0,\ 2\pi,\ 4\pi,\ \ldots\}$

are maxima of $\cos\alpha$

minima: $\cos(2k\cdot\pi+1) = -1$ with $k \in \mathbb{Z}$

i.e. $\{\ldots,\ -3\pi, -\pi,\ \pi,\ 3\pi,\ 5\pi,\ \ldots\}$

are minima of $\cos\alpha$

Inflection Points

$\sin\alpha$: $k \cdot \pi$

i.e. $\{\ldots,\ -2\pi,\ -\pi,\ 0,\ \pi,\ 2\pi,\ \ldots\}$

There is a concave/convex inflection point at π which switches left and right to convex/concave, then back again to concave/convex.

$\cos\alpha$: $\frac{\pi}{2} + k \cdot \pi$

i.e. $\left\{\ldots,\ -\frac{5}{2}\pi,\ \frac{\pi}{2} - \pi,\ \frac{\pi}{2},\ \frac{\pi}{2} + \pi,\ \frac{5}{2}\pi,\ \ldots\right\}$

There is a concave/convex inflection point at $\frac{\pi}{2}$ which switches left and right to convex/concave, then back again to concave/convex.

$\tan\alpha$: $k \cdot \pi$

i.e. $\{\ldots,\ -2\pi,\ -\pi,\ 0,\ \pi,\ 2\pi,\ \ldots\}$

All inflection points are concave/convex.

$\cot\alpha$: $\frac{\pi}{2} + k \cdot \pi$

i.e. $\left\{\ldots,\ -\frac{5}{2}\pi,\ -\frac{3}{2}\pi,\ \frac{\pi}{2},\ \frac{3}{2}\pi,\ \frac{5}{2}\pi,\ \ldots\right\}$

All inflection points are convex/concave.

Asymptotes

$\tan\alpha$: $(2k+1)\cdot\frac{\pi}{2}$

i.e. $\left\{\ldots,\ -\frac{3}{2}\pi,\ -\frac{1}{2}\pi,\ \frac{\pi}{2},\ \frac{3}{2}\pi,\ \frac{5}{2}\pi,\ \ldots\right\}$

$\cot\alpha$: $k\cdot\pi$

i.e. $\{\ldots,\ -2\pi,\ -\pi,\ 0,\ \pi,\ 2\pi,\ \ldots\}$

All asymptotes at $\tan\alpha$ and $\cot\alpha$ run vertically, i.e. are vertical asymptotes.

Trigonometric Values for Common Angles[1]

$\alpha(^\circ)$	α (rad)	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
0°	0	0	1	0	$\pm\infty$
15°	$\frac{\pi}{12}$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$2-\sqrt{3}$	$2+\sqrt{3}$
18°	$\frac{\pi}{10}$	$\frac{1}{4}(\sqrt{5}-1)$	$\frac{1}{4}\sqrt{10+2\sqrt{5}}$	$\frac{1}{5}\sqrt{25-10\sqrt{5}}$	$\sqrt{5+2\sqrt{5}}$
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$
36°	$\frac{\pi}{5}$	$\frac{1}{4}\sqrt{10-2\sqrt{5}}$	$\frac{1}{4}(1+\sqrt{5})$	$\sqrt{5-2\sqrt{5}}$	$\frac{1}{5}\sqrt{25+10\sqrt{5}}$
45°	$\frac{\pi}{4}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1
54°	$\frac{3\pi}{10}$	$\frac{1}{4}(1+\sqrt{5})$	$\frac{1}{4}\sqrt{10-2\sqrt{5}}$	$\frac{1}{5}\sqrt{25+10\sqrt{5}}$	$\sqrt{5-2\sqrt{5}}$
60°	$\frac{\pi}{3}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$
72°	$\frac{2\pi}{5}$	$\frac{1}{4}\sqrt{10+2\sqrt{5}}$	$\frac{1}{4}(\sqrt{5}-1)$	$\sqrt{5+2\sqrt{5}}$	$\frac{1}{5}\sqrt{25-10\sqrt{5}}$
75°	$\frac{5\pi}{12}$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$2+\sqrt{3}$	$2-\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	$\pm\infty$	0
108°	$\frac{3\pi}{5}$	$\frac{1}{4}\sqrt{10+2\sqrt{5}}$	$\frac{1}{4}(1-\sqrt{5})$	$-\sqrt{5+2\sqrt{5}}$	$-\frac{1}{5}\sqrt{25-10\sqrt{5}}$
120°	$\frac{2\pi}{3}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$
135°	$\frac{3\pi}{4}$	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1
180°	π	0	-1	0	$\pm\infty$
270°	$\frac{3\pi}{2}$	-1	0	$\pm\infty$	0
360°	2π	0	1	0	$\pm\infty$

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[1] Cf. Stratosphere Digital (2020): https://formula.amardesh.com/mathematics/trigonometric-functions-of-common-angles/, accessed 24 June 2021.

Phase Shifts of Trigonometric Functions

$\sin\left(x+\frac{\pi}{2}\right) = \cos x$ or $\sin(x+90^\circ) = \cos x$

$\cos\left(x+\frac{\pi}{2}\right) = -\sin x$ or $\cos(x+90^\circ) = -\sin x$

$\tan\left(x+\frac{\pi}{2}\right) = -\cot x$ or $\tan(x+90^\circ) = -\cot x$

$\cot\left(x+\frac{\pi}{2}\right) = -\tan x$ or $\cot(x+90^\circ) = -\tan x$

Relationships between Angle Functions

	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
$\sin\alpha$	$\sin\alpha$	$\pm\sqrt{1-\cos^2\alpha}$	$\frac{\tan\alpha}{\pm\sqrt{1+\tan^2\alpha}}$	$\frac{1}{\pm\sqrt{1+\cot^2\alpha}}$
$\cos\alpha$	$\pm\sqrt{1-\sin^2\alpha}$	$\cos\alpha$	$\frac{1}{\pm\sqrt{1+\tan^2\alpha}}$	$\frac{\cot\alpha}{\pm\sqrt{1+\cot^2\alpha}}$
$\tan\alpha$	$\frac{\sin\alpha}{\pm\sqrt{1-\sin^2\alpha}}$	$\frac{\pm\sqrt{1-\cos^2\alpha}}{\cos\alpha}$	$\tan\alpha$	$\frac{1}{\cot\alpha}$
$\cot\alpha$	$\frac{\pm\sqrt{1-\sin^2\alpha}}{\sin\alpha}$	$\frac{\cos\alpha}{\pm\sqrt{1-\cos^2\alpha}}$	$\frac{1}{\tan\alpha}$	$\cot\alpha$

Conversion for any Arbitrary Angle

	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
$90^\circ\pm\alpha$	$+\cos\alpha$	$\mp\sin\alpha$	$\mp\cot\alpha$	$\mp\tan\alpha$
$180^\circ\pm\alpha$	$\mp\sin\alpha$	$-\cos\alpha$	$\pm\tan\alpha$	$\pm\cot\alpha$
$270^\circ\pm\alpha$	$-\cos\alpha$	$\pm\sin\alpha$	$\mp\cot\alpha$	$\mp\tan\alpha$
$360^\circ\pm\alpha$	$\pm\sin\alpha$	$+\cos\alpha$	$\pm\tan\alpha$	$-\cot\alpha$

Periodicity of Trigonometric Functions

$$\left.\begin{aligned} \sin\,\alpha &= \sin\,(\alpha + k\cdot 2\pi)\\[2em] \cos\,\alpha &= \cos\,(\alpha + k\cdot 2\pi) \end{aligned}\right\}\ \text{period } 2\pi$$

$$\left.\begin{aligned} \tan\,\alpha &= \tan\,(\alpha + k\pi)\\[2em] \cot\,\alpha &= \cot\,(\alpha + k\pi) \end{aligned}\right\}\ \text{period } \pi$$

Symmetries in Trigonometric Functions

$$\sin(-x) = -\sin x$$

$$\cos(-x) = +\cos x$$

$$\tan(-x) = -\tan x$$

$$\cot(-x) = -\cot x$$

Goniometric Transformations

Sums and Differences $(\alpha \pm \beta)$

$$\sin(\alpha \pm \beta) = \sin\alpha \,\cos\beta \pm \cos\alpha \,\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha \,\cos\beta \mp \sin\alpha \,\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\,\alpha \pm \tan\,\beta}{1 \mp \tan\,\alpha\,\tan\,\beta} = \frac{\sin\,(\alpha \pm \beta)}{\cos\,(\alpha \pm \beta)}$$

$$\cot(\alpha \pm \beta) = \frac{\cot\,\alpha\,\cot\,\beta \mp 1}{\cot\,\beta\,\cot\,\alpha} = \frac{\cos\,(\alpha \pm \beta)}{\sin\,(\alpha \pm \beta)}$$

$$\sin\alpha + \sin\beta = 2\,\sin\frac{\alpha+\beta}{2}\,\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\,\cos\frac{\alpha+\beta}{2}\,\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\,\cos\frac{\alpha+\beta}{2}\,\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = 2\,\sin\frac{\alpha+\beta}{2}\,\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha \pm \sin\alpha = \sqrt{2}\,\sin(45^\circ \pm \alpha) = \sqrt{2}\,\cos(45^\circ \mp \alpha)$$

$$\tan\alpha \pm \tan\,\beta = \frac{\sin\,(\alpha \pm \beta)}{\cos\,\alpha\,\cos\,\beta}$$

$$\cot\alpha \pm \cot\beta = \frac{\sin\,(\alpha \pm \beta)}{\sin\,\alpha\,\sin\,\beta}$$

Double-Angle and Half-Angle Identities $\left(2\alpha; \frac{\alpha}{2}\right)$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha = \frac{2 \tan \alpha}{1 + \tan^2 \alpha}$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 = 2 \cos^2 \alpha - 1 = \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{2}{\cot \alpha - \tan \alpha}$$

$$\cot 2\alpha = \frac{\cot^2 \alpha - 1}{2 \cot \alpha} = \frac{\cot \alpha - \tan \alpha}{2}$$

$$\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\tan \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{1 - \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 + \cos \alpha}$$

$$\cot \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}} = \frac{1 + \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 - \cos \alpha}$$

Other Multiple-Angle Identities (n · α)

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

$$\sin 4\alpha = 8\sin\alpha\cos^3\alpha - 4\sin\alpha\cos\alpha$$

$$\sin 5\alpha = 16\sin\alpha\cos^4\alpha - 12\sin\alpha\cos^2\alpha + \sin\alpha$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$$

$$\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$$

$$\sin n\alpha = n\sin\alpha\cos^{n-1}\alpha - \binom{n}{3}\sin^3\alpha\cos^{n-3}\alpha + \binom{n}{5}\sin^5\alpha\cos^{n-5}\alpha + \dots$$

$$\cos n\alpha = \cos^n\alpha - \binom{n}{2}\sin^2\alpha\cos^{n-2}\alpha + \binom{n}{4}\sin^4\alpha\cos^{n-4}\alpha + \dots$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

$$\tan 4\alpha = \frac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}$$

$$\cot 3\alpha = \frac{\cot^3\alpha - 3\cot\alpha}{3\cot^2\alpha - 1}$$

$$\cot 4\alpha = \frac{\cot^4\alpha - 6\cot^2\alpha + 1}{4\cot^3\alpha - 4\cot\alpha}$$

Products $(\alpha \cdot \beta)$

$$\sin\alpha\,\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$$

$$\cos\alpha\,\cos\beta = \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta))$$

$$\sin\alpha\,\cos\beta = \frac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta))$$

$$\tan\alpha\,\tan\beta = \frac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta} = -\frac{\tan\alpha-\tan\beta}{\cot\alpha-\cot\beta}$$

$$\cot\alpha\,\cot\beta = \frac{\cot\alpha+\cot\beta}{\tan\alpha+\tan\beta} = -\frac{\cot\alpha-\cot\beta}{\tan\alpha-\tan\beta}$$

$$\tan\alpha\,\cot\beta = \frac{\tan\alpha+\cot\beta}{\cot\alpha+\tan\beta} = -\frac{\tan\alpha-\cot\beta}{\cot\alpha-\tan\beta}$$

Powers (α^n)

$$\sin^2 \alpha = \frac{1}{2}(1-\cos 2\,\alpha)$$

$$\cos^2 \alpha = \frac{1}{2}(1+\cos 2\,\alpha)$$

$$\tan^2 \alpha = \frac{1-\cos 2\,\alpha}{1+\cos 2\,\alpha}$$

$$\sin^3 \alpha = \frac{1}{4}(3\sin \alpha-\sin 3\,\alpha)$$

$$\cos^3 \alpha = \frac{1}{4}(3\cos \alpha+\cos 3\,\alpha)$$

$$\sin^4 \alpha = \frac{1}{8}(\cos 4\,\alpha-4\cos 2\,\alpha+3)$$

$$\cos^4 \alpha = \frac{1}{8}(\cos 4\,\alpha+4\cos 2\,\alpha+3)$$

$$\sin^5 \alpha = \frac{1}{16}(10\sin \alpha-5\sin 3\,\alpha+\sin 5\,\alpha)$$

$$\cos^5 \alpha = \frac{1}{16}(10\cos \alpha+5\cos 3\,\alpha+\cos 5\,\alpha)$$

$$\sin^6 \alpha = \frac{1}{32}(10-15\cos 2\,\alpha+6\cos 4\,\alpha-\cos 6\,\alpha)$$

$$\cos^6 \alpha = \frac{1}{32}(10+15\cos 2\,\alpha+6\cos 4\,\alpha+\cos 6\,\alpha)$$

Arcus Functions

Arcus functions (=cyclometric functions) are the inverse functions of the trigonometric functions (=angle functions/circular functions). However, this only applies to the so-called principal values, i.e. for certain co-domains, since the arcus functions are strictly monotonous and thus uniquely reversible only in certain intervals.

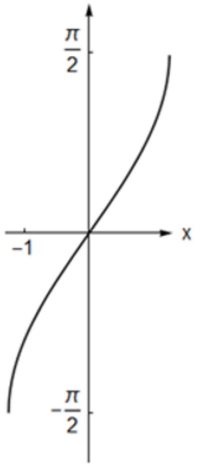
The following applies for the principal values:

$$\sin y = x \quad \text{with} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \quad \Leftrightarrow \quad y = \arcsin x$$

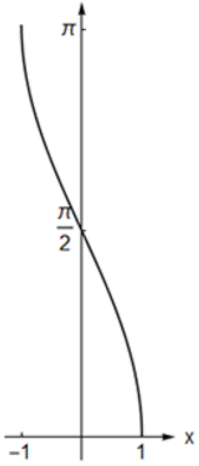
$$\cos y = x \quad \text{with} \quad 0 \leq y \leq \pi \quad \Leftrightarrow \quad y = \arccos x$$

$$\tan y = x \quad \text{with} \quad -\frac{\pi}{2} < y < \frac{\pi}{2} \quad \Leftrightarrow \quad y = \arctan x$$

$$\cot y = x \quad \text{with} \quad 0 < y < \pi \quad \Leftrightarrow \quad y = \operatorname{arccot} x$$



$y = \arcsin\ x$



$y = \arccos\ x$

Characteristics of Cyclometric Functions (in the principle values)

	$\arcsin x$	$\arccos x$	$\arctan x$	$\text{arccot}\, x$
Domain	$[-1;1]$	$[-1;1]$	$\mathbb{R}$	$\mathbb{R}$
Codomain	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[0,\pi]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[0,\pi]$
Zeros	0	1	0	-
Extremes	-	-	-	-
Inflection points	0	0	0	0
Asymptotes	-	-	$y=\frac{\pi}{2} \wedge y=\frac{-\pi}{2}$	$y=0 \wedge y=\pi$

Particular Function Values

$\arcsin(0) = \arctan(0) = 0$ $\arccos(0) = \frac{\pi}{2}$ $\arcsin(\pm 1) = \pm\frac{\pi}{2}$

$\arccos(1) = 0$ $\arccos(-1) = \pi$ $\arctan(\pm 1) = \pm\frac{\pi}{4}$

Arcus Functions of Negative x-Values

$\arcsin(-x) = -\arcsin x$

$\arccos(-x) = \pi - \arccos x$

$\arctan(-x) = -\arctan x$

$\text{arccot}(-x) = \pi - \text{arccot}\, x$

Hyperbolic Function

$\sinh x = \text{sh } x = \dfrac{e^x - e^{-x}}{2}$ read as “hyperbolic sine” (“sine hyperbolicus”)

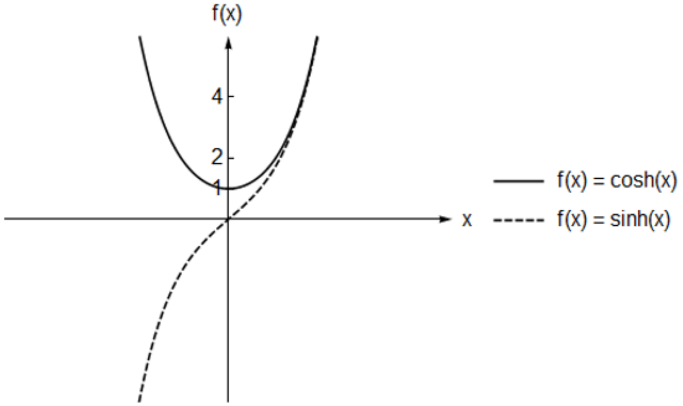
$\cosh x = \text{ch } x = \dfrac{e^x + e^{-x}}{2}$

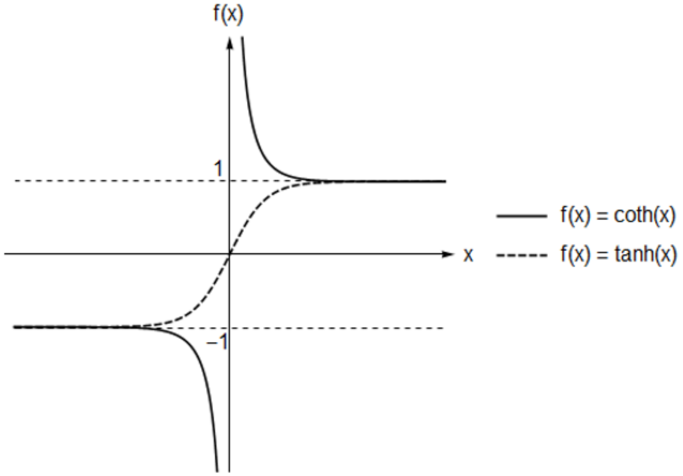
$\tanh x = \text{th } x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x} - 1}{e^{2x} + 1}$

$\coth x = \text{cth } x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \dfrac{e^{2x} + 1}{e^{2x} - 1}$ with $x \neq 0$

$\text{sech } x = \dfrac{2}{e^x - e^{-x}}$ read as “hyperbolic secant” (“secans hyperbolicus”)

$\text{csch } x = \dfrac{2}{e^x - e^{-x}}$ with $x \neq 0$ read as “hyperbolic cosecant” (“cosecans hyperbolicus”)





Hyperbolic Functions of Negative x-Values

$\sinh(-x) = -\sinh x$ $\tanh(-x) = -\tanh x$

$\cosh(-x) = \cosh x$ $\coth(-x) = -\coth x$

Periodicity of Hyperbolic Functions

$$\sinh x = \sinh(x + j2k\pi) \qquad \tanh x = \tanh(x + j2k\pi)$$

$$\cosh x = \cosh(x + j2k\pi) \qquad \coth x = \coth(x + j2k\pi)$$

Characteristics of Hyperbolic Functions[2]

	$\sinh x$	$\cosh x$	$\tanh x$	$\coth x$
Domain	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
Codomain	$\mathbb{R}$	$(1;\infty)$	$(-1;1)$	$(-\infty;-1)\vee(1;\infty)$
Zeros	0	-	0	-
Extremes	-	$x_{min} = 0$	-	-
Inflection points	0	-	0	-
Asymptotes	$y = e^{\frac{x}{2}}$ $y = -e^{-\frac{x}{2}}$	$y = e^{\frac{x}{2}}$ $y = -e^{-\frac{x}{2}}$	$y = 1$ $y = -1$	$x = 0^+$; $y = 1$ $x = 0^-$; $y = -1$

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[2] Depiction in reference to Bartsch, H. (2004), p. 414.

Particular Interconnections

$$\sinh x + \cosh x = e^{x}$$

$$\sinh x - \cosh x = e^{-x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

(hyperbolic Pythagorean theorem)

$$\coth x = \frac{1}{\tanh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{\tanh x}{\sinh x}$$

$$\operatorname{csch} x = \frac{\coth x}{\cosh x}$$

$$\operatorname{sech}^2 x + \tanh^2 x = 1$$

$$\coth^2 x - \operatorname{csch}^2 x = 1$$

Sums and Differences $(x_1 \pm x_2)$

$$\sinh\, x_1 \pm x_2 = \sinh\, x_1 \cosh\, x_2 \pm \cosh\, x_1 \sinh\, x_2$$

$$\cosh\, x_1 \pm x_2 = \cosh\, x_1 \cosh\, x_2 \pm \sinh\, x_1 \sinh\, x_2$$

$$\tanh\, x_1 \pm x_2 = \frac{\tanh\, x_1 \pm \tanh\, x_2}{1 \pm \tanh\, x_1 \tanh\, x_2}$$

$$\coth\, x_1 \pm x_2 = \frac{1 \pm \coth\, x_1 \coth\, x_2}{\coth\, x_1 \pm \coth\, x_2}$$

$$\sinh\, x_1 \pm \sinh\, x_2 = 2\ \sinh \frac{x_1 \pm x_2}{2} \cosh \frac{x_1 \mp x_2}{2}$$

$$\cosh\, x_1 + \cosh\, x_2 = 2\ \cosh \frac{x_1 + x_2}{2} \cosh \frac{x_1 - x_2}{2}$$

$$\cosh\, x_1 - \cosh\, x_2 = 2\ \sinh \frac{x_1 + x_2}{2} \sinh \frac{x_1 - x_2}{2}$$

$$\tanh\, x_1 \pm \tanh\, x_2 = \frac{\sinh\,(x_1 \pm x_2)}{\cosh\, x_1 \cosh\, x_2}$$

$$\coth\, x_1 \pm \coth\, x_2 = \frac{\sinh\,(x_1 \pm x_2)}{\sinh\, x_1 \sinh\, x_2}$$

Double-Argument and Half-Argument Identities $\left(2x; \frac{x}{2}\right)$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \sinh^2 x + \cosh^2 x = 2 \cosh^2 x - 1 = 2 \sinh^2 x + 1$$

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

$$\coth 2x = \frac{1 + \coth^2 x}{2 \coth x}$$

$$\sinh \frac{x}{2} = \sqrt{\frac{\cosh x - 1}{2}} \cdot \operatorname{sgn} x = \frac{\sinh x}{\sqrt{2\,(\cosh x + 1)}}$$

$$\cosh \frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}} = \frac{\sinh x}{\sqrt{2\,(\cosh x - 1)}}$$

$$\tanh \frac{x}{2} = \frac{\sinh x}{\cosh x + 1} = \frac{\cosh x - 1}{\sinh x} = \sqrt{\frac{\cosh x - 1}{\cosh x + 1}} \cdot \operatorname{sgn} x$$

$$\coth \frac{x}{2} = \frac{\sinh x}{\cosh x - 1} = \frac{\cosh x + 1}{\sinh x} = \sqrt{\frac{\cosh x + 1}{\cosh x - 1}} \cdot \operatorname{sgn} x$$

Other Multiple-Argument Identities $(n \cdot x)$

$$\sinh 3x = \sinh x\,(4\cosh^2 x - 1)$$

$$\sinh 4x = \sinh x \cosh x\,(8\cosh^2 x - 4)$$

$$\sinh 5x = \sinh x\,(1 - 12\cosh^2 x + 16\cosh^4 x)$$

$$\cosh 3x = \cosh x\,(4\cosh^2 x - 3)$$

$$\cosh 4x = 1 - 8\cosh^2 x + 8\cosh^4 x$$

$$\cosh 5x = \cosh x\,(5 - 20\cosh^2 x + 16\cosh^4 x)$$

$$\sinh nx = \binom{n}{1}\cosh^{n-1} x \sinh x + \binom{n}{3}\cosh^{n-3} x \sinh^3 x + \binom{n}{5}\cosh^{n-5} x \sinh^5 x + \ldots$$

$$\cosh nx = \cosh^n x + \binom{n}{2}\cosh^{n-2} x \sinh^2 x + \binom{n}{4}\cosh^{n-4} x \sinh^4 x + \ldots$$

Powers (x^n)

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$$

$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

$$\sinh^3 x = \frac{1}{4}(-3 \sinh x + \sinh 3x)$$

$$\cosh^3 x = \frac{1}{4}(3 \cosh x + \cosh 3x)$$

$$\sinh^4 x = \frac{1}{8}(3 - 4 \cosh 2x + \cosh 4x)$$

$$\cosh^4 x = \frac{1}{8}(3 + 4 \cosh 2x + \cosh 4x)$$

$$\sinh^5 x = \frac{1}{16}(-10 \sinh x + 5 \sinh 3x + \sinh 5x)$$

$$\cosh^5 x = \frac{1}{16}(10 \cosh x + 5 \cosh 3x + \cosh 5x)$$

$$\sinh^6 x = \frac{1}{32}(-10 + 15 \cosh 2x - 6 \cosh 4x + \cosh 6x)$$

$$\cosh^6 x = \frac{1}{32}(10 + 15 \cosh 2x + 6 \cosh 4x + \cosh 6x)$$

Binomials (De Moivre's Theorem)

$$(\cosh x \pm \sinh x)^n = \cosh nx \pm \sinh nx \qquad n = 2, 3, \ldots$$

Products $(x_1 \cdot x_2)$

$$\sinh x_1 \; \sinh x_2 = \frac{1}{2}(\cosh(x_1 + x_2) - \cosh(x_1 - x_2))$$

$$\cosh x_1 \; \cosh x_2 = \frac{1}{2}(\cosh(x_1 + x_2) + \cosh(x_1 - x_2))$$

$$\sinh x_1 \; \cosh x_2 = \frac{1}{2}(\sinh(x_1 + x_2) + \sinh(x_1 - x_2))$$

$$\tanh x_1 \; \tanh x_2 = \frac{\tanh x_1 + \tanh x_2}{\coth x_1 + \coth x_2}$$

Area Functions

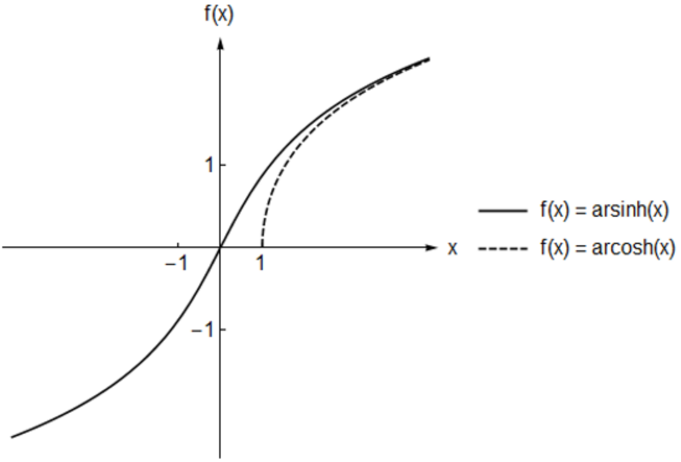
In the codomain of the hyperbolic functions, the area functions form the inverse functions of the corresponding hyperbolic functions.

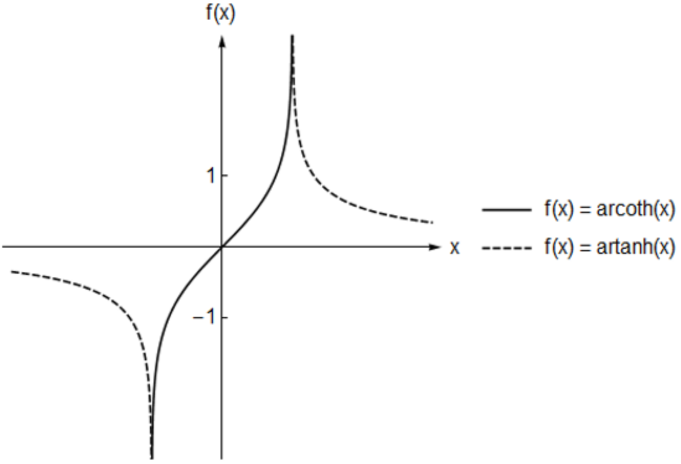
$y = \operatorname{arsinh} x \quad \Rightarrow \quad x = \sinh y$ read as “area hyperbolic sine”

$y = \operatorname{arcosh} x \quad \Rightarrow \quad x = \cosh y$

$y = \operatorname{artanh} x \quad \Rightarrow \quad x = \tanh y$

$y = \operatorname{arcoth} x \quad \Rightarrow \quad x = \coth y$





Area Functions of Negative x-Values

$$\begin{aligned}
\operatorname{arsinh}(-x) &= -\operatorname{arsinh} x\\
\operatorname{artanh}(-x) &= -\operatorname{artanh} x\\
\operatorname{arcoth}(-x) &= -\operatorname{arcoth} x
\end{aligned}$$

Characteristics of Area Functions [3]

	arsinh x	arcosh x	artanh x	arcoth x
Domain	$\mathbb{R}$	$(1;\infty)$	$(-1;1)$	$(-\infty;-1) \vee (1;\infty)$
Codomain	$\mathbb{R}$	$(0;\infty)$	$\mathbb{R}$	$\mathbb{R}^*$
Zeros	0	1	0	-
Inflection points	0	-	0	-
Asymptotes	$y = \ln 2x$ $y = -\ln(-2x)$	$y = \ln 2x$	$x = 1$ $x = -1$	$y = 0^-\,; x = -1$ $y = 0^+\,; x = 1$

Sums and Differences $(x_1 + x_2)$

$$\text{arsinh } x_1 \pm \text{arsinh } x_2 = \text{arsinh}\left(x_1\sqrt{1+x_2^2} \pm x_2\sqrt{1+x_1^2}\right)$$

$$\text{arcosh } x_1 \pm \text{arcosh } x_2 = \text{arcosh}\left(x_1x_2 \pm \sqrt{(x_1^2-1)(x_2^2-1)}\right)$$

$$\text{artanh } x_1 \pm \text{artanh } x_2 = \text{artanh } \frac{x_1 \pm x_2}{1 \pm x_1x_2}$$

$$\text{arcoth } x_1 \pm \text{arcoth } x_2 = \text{arcoth } \frac{1 \pm x_1x_2}{x_1 \pm x_2}$$

---

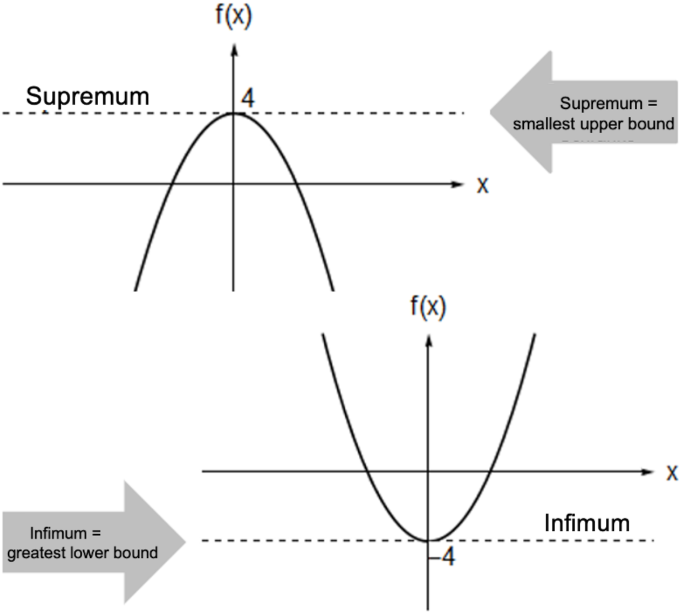
[3] Cf. Bartsch, H. 2004, p. 419.

9.3 Characteristics of Real Functions

9.3.1 Boundedness

If $C_f = \mathbb{R}$, the (real) function f is unconstrained; if $C_f \subset \mathbb{R}$, it is constrained.

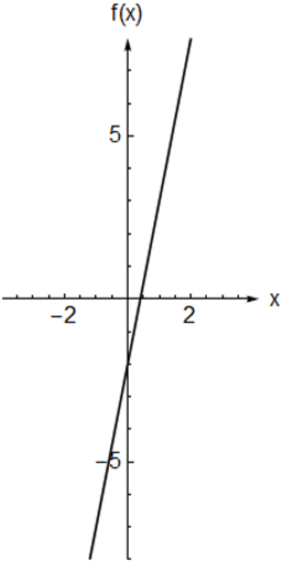
upper bound	**lower bound**
if $f(x) \leq u$ and $u \in \mathbb{R}$	if $f(x) \geq l$ and $l \in \mathbb{R}$
$\rightarrow$ all elements are smaller/ equal than the upper bound	$\rightarrow$ all elements are bigger/ equal than the lower bound
$f(x) = -x^2 + 4$	$f(x) = x^2 - 4$



unbounded

The function has no constraints.

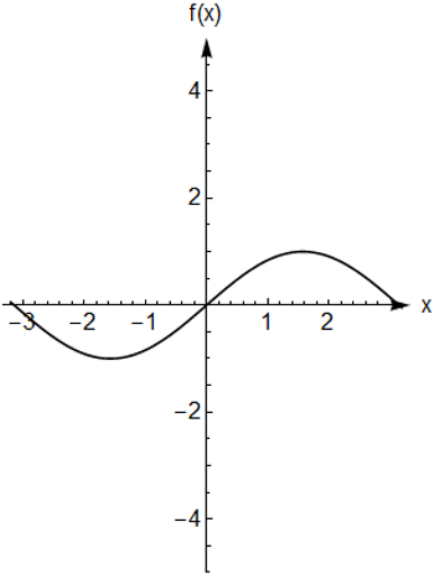
Example: $f(x) = 5x - 2$



bounded from below and above

$l \leq f(x) \leq u$ with $u, l \in \mathbb{R}$

Example: $f(x) = \sin(x)$



9.3.2 Symmetry

9.3.2.1 Axial Symmetry

Axial Symmetry to the y-Axis

If a function $f(x)$ is axially symmetric to the y-axis, the graph of the function y is mirrored at the y-axis:

$f(-x) = f(x)$ for all $x \in D_f$

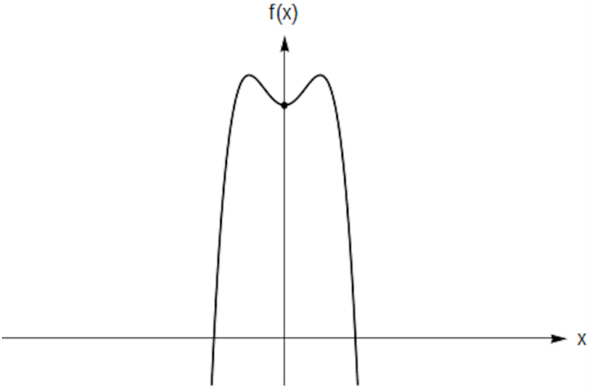
The x-values of the function $f(x)$, which are axially symmetric to the y-axis, only have even exponents.

<u>Example:</u>

Prove that the function $f(x) = -7x^4 + 6x^2 + 10$ is axially symmetric to the y-axis.

$$\begin{array}{lrcl} \text{Approach:} & f(-x) & = & f(x) \\ \Rightarrow & f(-x) & = & -7(-x)^4 + 6(-x)^2 + 10 = -7x^4 + 6x^2 + 10 \end{array}$$

$\rightarrow$ The function $f(x) = -7x^4 + 6x^2 + 10$ is axially symmetric to the y-axis.



Axial Symmetry to any Arbitrary Line with $x = x_0$

Axial symmetry to a straight line, on which all x-values receive the value x_0, is present if the following applies:

$f(x_0+h) = f(x_0-h)$ with $h \in \mathbb{R}$ and $h > 0$.

Example:

Prove that the function $f(x) = x^2+6x+9$ is axially symmetric to the straight line $x_0 = -3$.

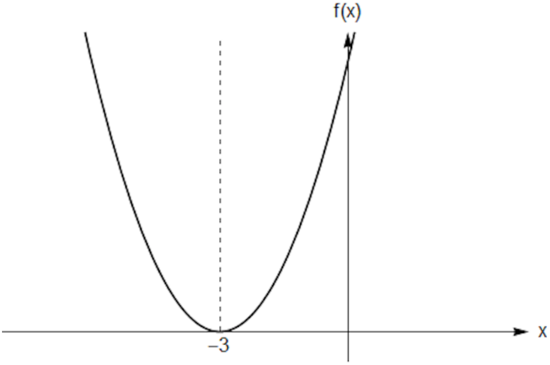
Insert (x_0+h) into the function:
$$\begin{aligned} f(-3+h) &= (-3+h)^2+6(-3+h)+9 \\ &= 9-6h+h^2-18+6h+9 \\ &= h^2 \end{aligned}$$

Insert (x_0-h) into the function:
$$\begin{aligned} f(-3-h) &= (-3-h)^2+6(-3-h)+9 \\ &= 9+6h+h^2-18-6h+9 \\ &= h^2 \end{aligned}$$

The following applies:

$f(-3+h) = f(-3-h)$

→ The function
$f(x) = x^2+6x+9$
is therefore axially
symmetric to the
straight line $x_0 = -3$.



9.3.2.2 Point Symmetry

Point Symmetry to the Point of Origin

If a function $f(x)$ is symmetric to the point of origin, it is mirrored at the point of origin of the coordinate system.

The following applies: $f(-x) = -f(x)$ for all $x \in D_f$

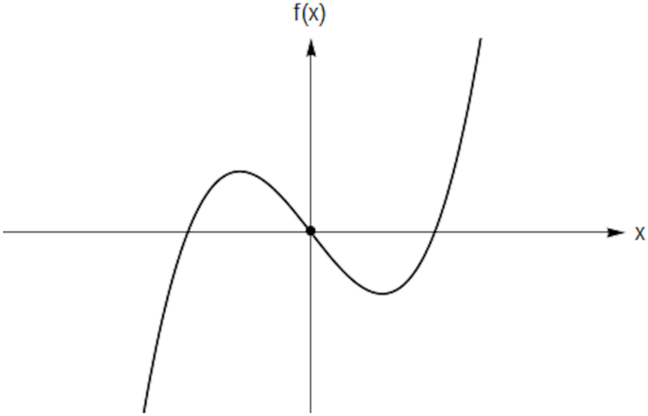
$f(x)$ has only odd exponents.

Example:

Prove that the function $f(x) = 3x^3 - 2x$ is point symmetric to the point of origin.

Approach:
$$\begin{aligned} f(-x) &= -f(x) \\ 3(-x)^3 - 2(-x) &= -(3x^3 - 2x) \\ \Rightarrow -3x^3 + 2x &= -3x^3 + 2x \end{aligned}$$

→ The function $f(x) = 3x^3 - 2x$ is therefore point symmetric to the point of origin.

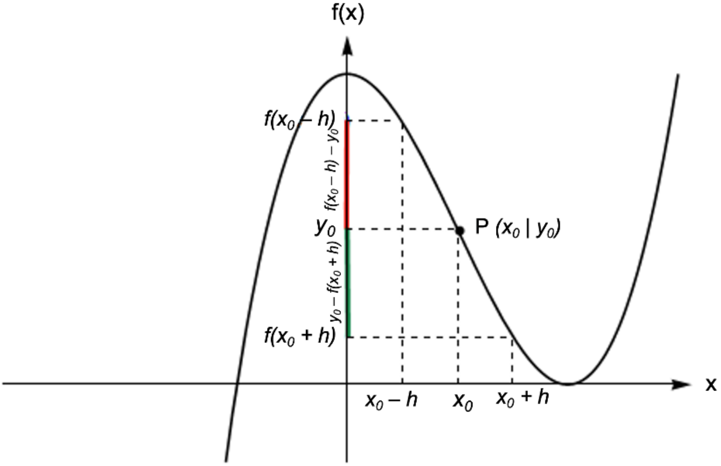


Point Symmetry to any Arbitrary Point

Point symmetry to any arbitrary point $(x_0|y_0)$ exists if the following applies:

$$f(x_0+h)-y_0 = y_0 - f(x_0-h).$$

x_0 and y_0 are the coordinates of the point of symmetry.



Example:

Is the function $f(x) = x^3 + 3x^2$ symmetric to the point $P(-1 \mid 2)$?

Calculate $f(x_0+h) - y_0$:

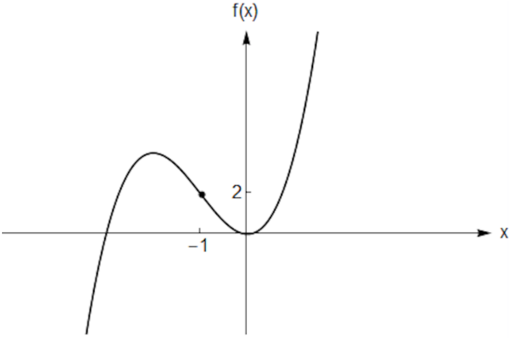
$$\begin{aligned} f(x_0+h)-y_0 &= \left[(-1+h)^3+3(-1+h)^2\right]-2 \\ &= \left[(1-2h+h^2)\cdot(-1+h)+3(1-2h+h^2)\right]-2 \\ &= \left[-1+2h-h^2+h-2h^2+h^3+3-6h+3h^2\right]-2 \\ &= \left[h^3-3h+2\right]-2 \\ &= h^3-3h \end{aligned}$$

Calculate $y_0 - f(x_0-h)$:

$$\begin{aligned} y_0-f(x_0-h) &= 2-\left[(-1-h)^3+3(-1-h)^2\right] \\ &= 2-\left[(1+2h+h^2)\cdot(-1-h)+3(1+2h+h^2)\right] \\ &= 2-\left[-1-2h-h^2-h-2h^2-h^3+3+6h+3h^2\right] \\ &= 2-\left[-h^3+3h+2\right] \\ &= h^3-3h \end{aligned}$$

Compare the results of both steps:

→ The function $f(x) = x^3 + 3x^2$ is symmetric to the point $P(-1 \mid 2)$ since the following applies: $f(x_0+h) - y_0 = y_0 - f(x_0-h)$ or $h^3 - 3h = h^3 - 3h$



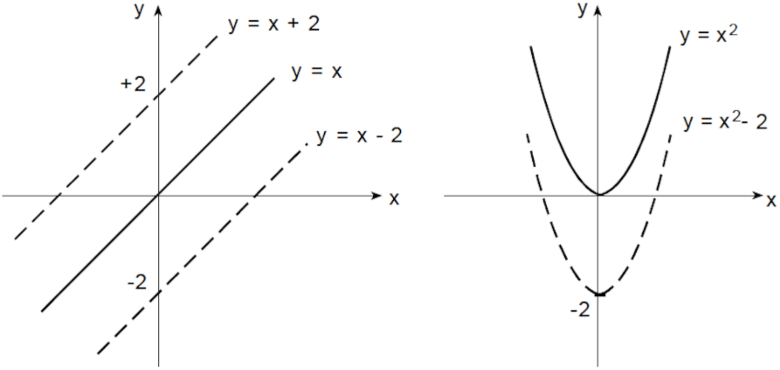
9.3.3 Transformations

- **Shift**

If a function $y = y(x)$ is shifted by b towards the y-axis, the original function $y = y(x)$ is transformed to a function $g = g(x)$ as follows:

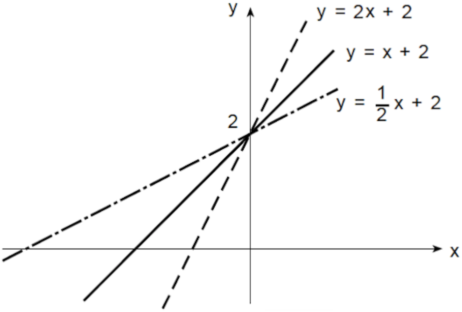
$g(x) = y(x) + b$ with $b \in \mathbb{R}$

b = change of the absolute term



If a linear function $y = y(x)$ is rotated at the intersection with the y-axis (ordinate) by a constant factor c, the original function $y = y(x)$ is transformed to a function $g = g(x)$ as follows:

$g(x) = y(c \cdot x)$ with $c \in \mathbb{R}$ (c = change of the slope)



• Stretch/Compression

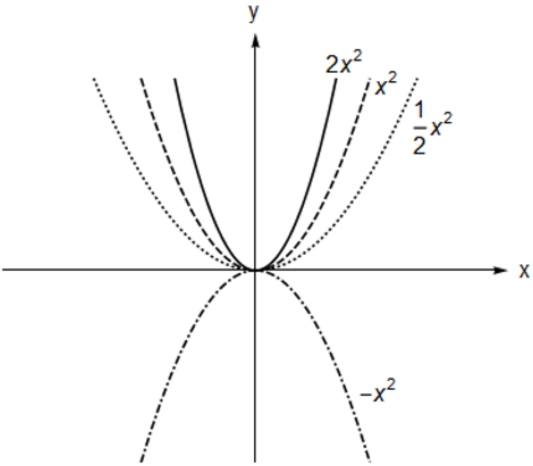
If a quadratic function $y = y(x)$ is d-foldly stretched/compressed, the original function $y = y(x)$ is transformed to a function $g = g(x)$ as follows:

$g(x) = d \cdot y(x)$ with $b \in \mathbb{R}$

$|\, d \,| \;\; > 1$ stretch towards the y-axis

$|\, d \,| \;\; < 1$ compression towards the y-axis

$d \;\; < 0$ reflection across the x-axis with stretch/compression



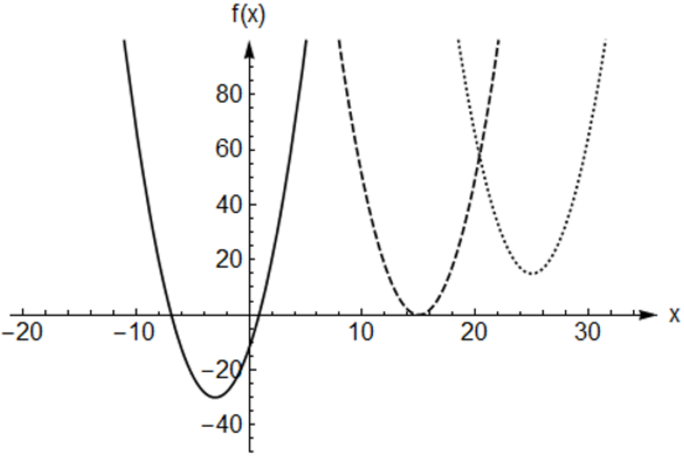
9.3.3.1 Vertex Form

General form $ax^2 + bx + c = f(x)$

Vertex form $a \cdot (x - d)^2 + e = f(x)$

Quadratic functions have either a maximum or a minimum. This turning point is the vertex. Quadratic functions have an axis of symmetry. It runs parallel to the y-axis through the vertex. Quadratic functions have either none, one or two zeros. These zeros can be determined using the p/q formula.

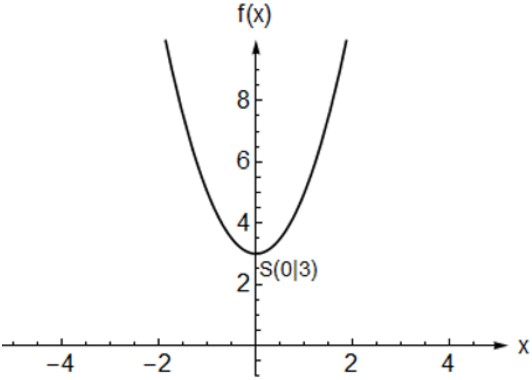
This is as follows: $x_{1,\,2} = -\frac{p}{2} \pm \sqrt{(\frac{p}{2})^2 - q}$



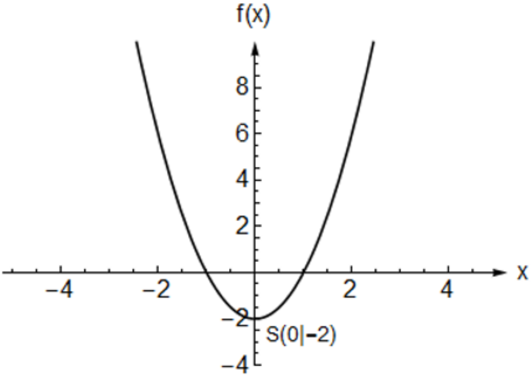
Shift along the y-axis:

$f(x) = a \cdot (x - d)^2 + e$

For $e > 0$, the parabola is shifted upwards along the y-axis by e units.



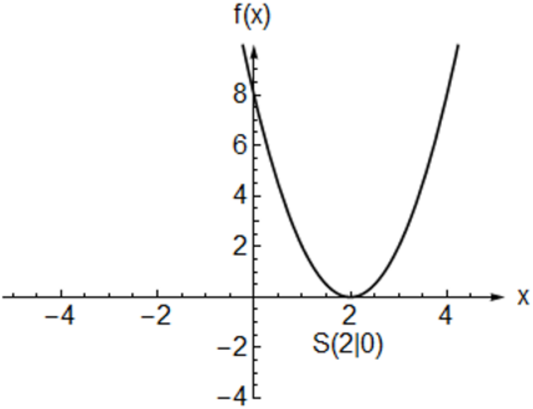
For $e < 0$, the parabola is shifted downwards along the y-axis by e units.



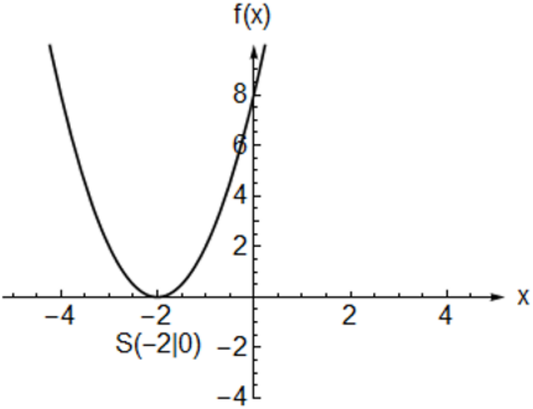
Shift along the x-axis:

$f(x) = a \cdot (x - d)^2 + e$

For $d > 0$, the parabola is shifted to the right along the x-axis by d units.



For $d < 0$, the parabola is shifted to the left along the x-axis by d units.



9.3.4 Continuity

If a function f is differentiable in x_0, it is continuous there.

Discontinuities

See also Chapter 9.2.

There are three types of discontinuities:

- infinite discontinuities (poles),
- removable discontinuities (singularities),
- jump discontinuities.

9.3.5 Infinite Discontinuities

An infinite discontinuity (pole) of a broken rational function $f = f(x)$ at x_0 always exists when the denominator is equal to zero (singularity):

$$\lim_{x \to x_0} f(x) = \pm\infty$$

A pole cannot be removed.

Example:

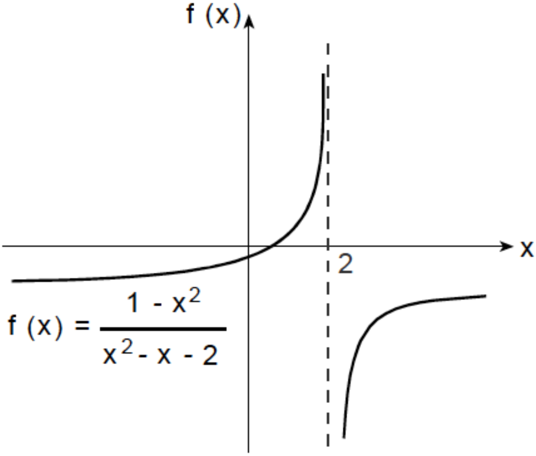
$$f(x) = \frac{1-x^2}{x^2-x-2} = \frac{(1+x)\cdot(1-x)}{(x-2)\cdot(x+1)} = \frac{(1-x)}{(x-2)}$$

$$\Rightarrow \qquad D_f = \mathbb{R}\setminus\{-1;2\}$$

The singularity at $x = 2$ cannot be removed; i.e. it cannot be simplified. There is a pole here.

$$\lim_{x\to 2^-} f(x) \approx f(1.9999) = +\infty$$

$$\lim_{x\to 2^+} f(x) \approx f(2.0001) = -\infty$$



9.3.6 Removable Discontinuities

A removable discontinuity (singularity) of a broken rational function $f = f(x)$ at x_0 always exists when the numerator and denominator simultaneously equal zero (singularity). The discontinuities can be removed by assigning the limit $\lim\limits_{x \to x_0} f(x)$ to the discontinuities.

Example: see above

$$f(x) = \frac{1-x^2}{x^2-x-2} = \frac{(1+x)\cdot(1-x)}{(x-2)\cdot(x+1)} = \frac{(1-x)}{(x-2)}$$

$$\Rightarrow \qquad D_f = \mathbb{R} \setminus \{-1;2\}$$

The singularity at $x = -1$ can be removed; i.e. it can be simplified. The corresponding limit exists:

$$\lim_{x \to -1^-} f(x) = -\frac{2}{3}; \quad \lim_{x \to -1^+} f(x) = -\frac{2}{3} \quad \Rightarrow$$

There is a removable discontinuity here.

Remark:

In the original function $f(x)$, the discontinuity remains by definition a singularity, which must be excluded from the domain.

9.3.7 Jump Discontinuities

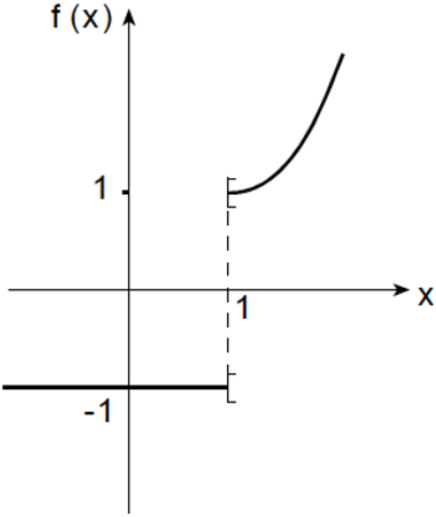
At a jump discontinuity x_0 of a function $f = f(x)$, the left and right limits are distinct:

$$\lim_{x\to x_0^-} f(x) \quad \neq \quad \lim_{x\to x_0^+} f(x) \underbrace{(\neq \pm\infty)}_{\text{cf. pole}}$$

Example:

$$f(x) = \begin{cases} -1 & \text{for } x < 1 \\ x^2 & \text{for } x \geq 1 \end{cases}$$

$\Rightarrow$ jump discontinuity at $x = 1$



9.3.8 Homogeneity

A function f with n independent variables $x_1,\ldots,x_n$; $f = f(x_1,\ldots,x_n)$ with $D_f \subseteq \mathbb{R}^n$ is homogeneous of degree r if $\lambda \geq 0$ applies to each real number:

$f(\lambda x_1,\ldots,x_n) = \lambda^r \cdot f(x_1,\ldots,x_2) \cdot r =$ degree of homogeneity

Examples:

$$f(x_1,x_2) = x_1^2 + 4x_1x_2 + 5x_2^2$$

$$\Rightarrow \quad f(\lambda x_1,\lambda x_2) = (\lambda x_1)^2 + 4\lambda x_1\lambda x_2 + 5(\lambda x_2)^2$$

$$= \lambda^2 x_1^2 + \lambda^2 4x_1x_2 + \lambda^2 5x_2^2$$

$$= \lambda^2(x_1^2 + 4x_1x_2 + 5x_2^2)$$

$$= \lambda^2 \cdot f(x_1,x_2)$$

$\Rightarrow$ f is homogeneous of degree 2.

9.3.9 Periodicity

A function f with $f = f(x)$ is periodic with a period T if the following applies:

$$f(x) = f(x \pm nT) \quad \text{with} \quad (x \pm nT) \in D_f \quad n \in \mathbb{Z}^* \quad T > 0$$

The smallest period is also called the (primitive) period of f.

Example:

The sine function $f(x) = \sin x$ is a periodic function with the period $T = 2\pi$:

$$\sin x = \sin(x + k2\pi) \quad \text{with} \quad k \in \mathbb{R}$$

9.3.10 Zeros

Zero is the intersection of a function $f(x)$ with the x-axis.
$\Rightarrow \; f(x) = 0$

Example:

$$f(x) = x^2 - x - 6 = 0$$

$\Rightarrow$ p/q formula:

$$x_1/x_2 = +\frac{1}{2} \pm \sqrt{\left(-\frac{1}{2}\right)^2 + 6}$$

$$x_1 = 0.5 + \sqrt{6.25} = 3$$

$$x_2 = 0.5 - \sqrt{6.25} = -2$$

9.3.11 Local Extremes

necessary condition: $f'(x) \overset{!}{=} 0$

sufficient condition: $f''(x) > 0 \Rightarrow$ minimum

$f''(x) < 0 \Rightarrow$ maximum

Example:

$$f(x) = x^3 - 8x^2 + 8x - 3$$

$$\Rightarrow \qquad f'(x) = 3x^2 - 16x + 8 = 0$$

$$f'(x) = x^2 - 5.\bar{3}x + 2.6667 = 0$$

$\Rightarrow$ p/q formula: $x_1, x_2 = \frac{5.\bar{3}}{2} \pm \sqrt{\left(-\frac{5.\bar{3}}{2}\right)^2 - 2.6667}$

$$x_1 = 2.6667 + \sqrt{4.4444} = 4.7749$$

$$x_2 = 2.6667 - \sqrt{4.4444} = 0.5585$$

$$y_1 = f(4.7749) = x^3 - 8x^2 + 8x - 3 = -38.3320$$

$$y_2 = f(0.5585) = x^3 - 8x^2 + 8x - 3 = -0.8532$$

The function has extremes at $P_1(4.7749 \mid -38.3320)$ and $P_2(0.5585 \mid -0.8532)$.

$\Rightarrow$ $f''(4.7749) = 6x - 16 = 12.649 > 0$ $\Rightarrow$ minimum $P_1(4.7749 \mid -38.3320)$

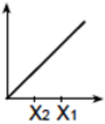
$\Rightarrow$ $f''(0.5585) = 6x - 16 = -12.649 < 0$ $\Rightarrow$ maximum $P_1(0.5585 \mid -0.8532)$

9.3.12 Monotonicity

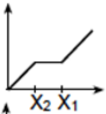
The monotonicity defines the slope of a function
(monotonic $\widehat{=}$ uniform).

Applies to all $x_1, x_2 \in I$ ($I =$ interval) with $x_2 > x_1$ and $I \in D_f$:

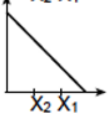
$f(x_1) > f(x_2) \quad \Rightarrow \quad f$ is strictly monotonically increasing



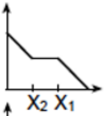
$f(x_1) \geq f(x_2) \quad \Rightarrow \quad f$ is monotonically increasing



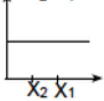
$f(x_1) < f(x_2) \quad \Rightarrow \quad f$ is strictly monotonically decreasing



$f(x_1) \leq f(x_2) \quad \Rightarrow \quad f$ is monotonically decreasing

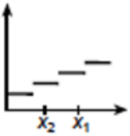


$f(x_1) = f(x_2) \quad \Rightarrow \quad f$ runs parallel to the x-axis
$\Rightarrow$ the slope is always zero
$\Rightarrow$ the monotonicity is always constant



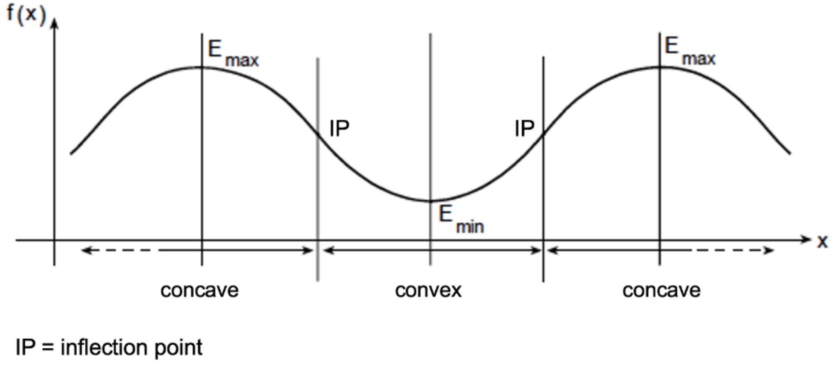
$\Rightarrow$ The monotonic behaviour changes at the extremes ($=$ minimum or maximum).

$\Rightarrow$ The monotonicity has no relation to the continuity; a step function is for example also monotonically increasing/decreasing.



$\Rightarrow$ There is only one slope between two points; not at one point.

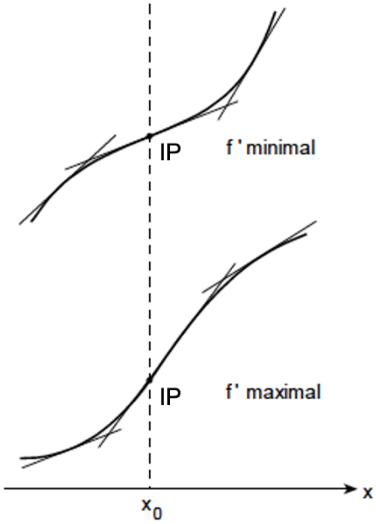
9.3.13 Concavity and Convexity | Inflection Points



Forms of Curvature

- **Convexity** exists if the second derivative of a function is greater than zero. $(f''(x) > 0)$
- **Concavity** exists if the second derivative of a function is smaller than zero. $(f''(x) < 0)$

At the inflection points (IP), the curvature behaviour of a function changes from concave to convex or from convex to concave.



- concave/convex inflection point:

  at the point where the slope is negative, the slope is smallest

  $f'(x) =$ minimal

  $f''(x) = 0$

  $f'''(x) > 0$

- convex/concave inflection point:

  at the point where the slope is positive, the slope is greatest

  $f'(x) =$ maximal

  $f''(x) = 0$

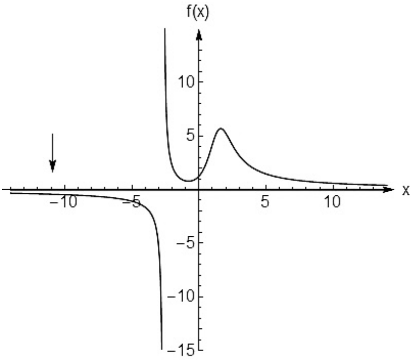
  $f'''(x) < 0$

9.3.14 Asymptotes

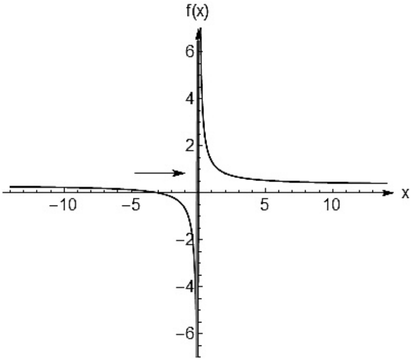
An asymptote is a function that approaches another function without intersecting or touching it.

There are four different types of asymptotes:

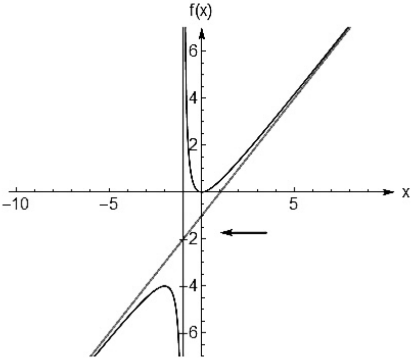
horizontal asymptote



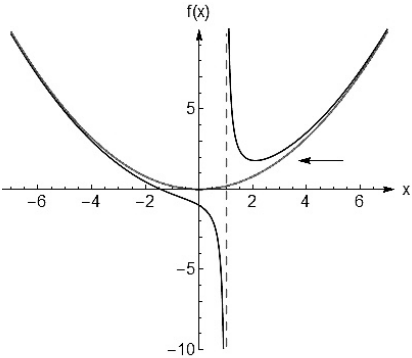
vertical asymptote



oblique asymptote



asymptotic curve



9.3.14.1 Horizontal Asymptotes

Determination of the asymptote using limiting behaviour (in Latin: limes) or by comparing the degree of the numerator (n) and the degree of the denominator (m).

degree of the numerator $=$

degree of the denominator

$$f(x) = \frac{1x^4 - 3}{4 + 1x^4}$$

Consideration of the highest exponents. The coefficients in front of the bases with the highest exponents determine the level of horizontal asymptotes.

Equation of asymptotes:

$g(x) = 1$

degree of the numerator <

degree of the denominator

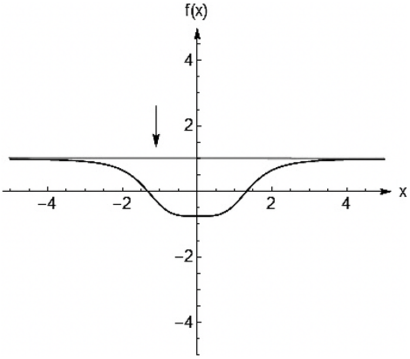
$$f(x) = \frac{x^1 - 5}{x^2}$$

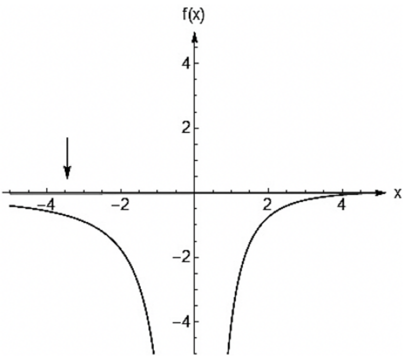
Consideration of the highest exponents. The quotient of the highest exponents - here $0/1$ - is zero. Thus the horizontal asymptote is equal to the x-axis.

Equation of asymptotes:

$g(x) = 0$

(The line is on the x-axis)





Example:

$$f(x) = \frac{3x^2 - 1}{x + 5x^2}$$

Division by x with the highest exponent, here divided by x^2

$$\Rightarrow \qquad f(x) = \frac{3 - \frac{1}{x^2}}{5 + \frac{1}{x^1}}$$

Forming the limits towards $\pm \to \infty$ $\qquad \frac{\text{number}}{\text{infinity}} \approx 0$

$$\lim_{x \to \pm\infty} f(x) = \frac{3 - \frac{1}{x^2}}{5 + \frac{1}{x^1}}$$

$$\lim_{x \to \pm\infty} f(x) = \frac{3 - 0}{5 + 0}$$

Equation of asymptotes:

$$g(x) = \frac{3}{5}$$

9.3.14.2 Vertical Asymptote

Asymptotes are determined by the poles.
(pole $=$ vertical line)

$$\frac{p(x) \neq 0}{q(x) = 0}$$

<u>Example:</u>

$$f(x) = \frac{x^2 - x - 2}{(x-3)(x+5)}$$

1. Determine the singularity:

$(x-3)(x+5) = 0$

$x_1 = 3\ ; x_2 = -5$

2. Insert the singularities into the numerator:

$f(3) = 3^2 - 3 - 2 = 4$

$f(-5) = (-5)^2 + 5 - 2 = 28$

$\rightarrow$ both numerators $\neq 0 \rightarrow$ poles are at 3 and at -5.

9.3.14.3 Oblique Asymptote

An oblique asymptote is a straight line with a slope $\neq 0$.

Note the degree of the numerator and the degree of the denominator:
degree of the numerator $=$ degree of the denominator $+1$

e.g.: $f(x) = \dfrac{x^2}{x^1 + 1}$

Equation of asymptotes: $g(x) = mx + b$

Approach: Step 1: determine the degree of the numerator and of the denominator

Step 2: polynomial division

Step 3: observation of the limit

Step 1: $\dfrac{x^2}{x^1 + 1}$

Step 2: $(x^2 + 0x^1) \div (x + 1) = x - 1 + \frac{1}{x+1}$

Step 3: $\lim\limits_{x \to \pm\infty} (\frac{1}{x+1}) = 0 \quad \Rightarrow \quad g(x) = x - 1$ (equation of asymptotes)

9.3.14.4 Asymptotic Curve

The asymptotic curve is a function that is not a straight line (oblique asymptote) but describes a curved function (curve).

Note the degree of the numerator and the degree of the denominator:
degree of the numerator > degree of the denominator $+1$

e.g.: $f(x) = \dfrac{x^4 - 1}{x}$

Approach: Step 1: determine the degree of the numerator and of the denominator

Step 2: polynomial division

Step 3: observation of the limit

Step 1: $\dfrac{x^4 - 1}{x}$

Step 2: $(x^4 + 0x^2 - 1) \div (x) = x^3 - \dfrac{1}{x}$

Step 3: $\lim\limits_{x \to \pm\infty} (\tfrac{1}{x}) = 0 \quad \Rightarrow \quad g(x) = x^3$ (equation of asymptotes)

Key rules

denominator $= 0$ and numerator $\neq 0$	vertical asymptote $=$ pole
$n < m$ or $n = m$	horizontal asymptote
$n = m + 1$	oblique asymptote
$n > m + 1$	asymptotic curve

9.3.15 Tangent Lines to a Curve

A tangent line to a curve is a straight line that touches a function $f(x)$ at a point P_0. The slope of the tangent m_{tan} describes the slope of the function $f(x)$ at a point P_0 or at a position x_0.
$\rightarrow m_{tan} = f'(x_0)$

Example:

given: $f(x) = 3x^2 + 1$ $\qquad x_0 = 1$

to be found: $y = m \cdot x + b$ $\qquad$ (equation of tangent line)

1. Determine the derivative of $f(x)$

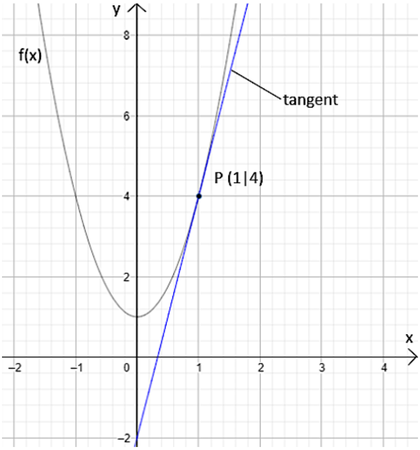
$$\Rightarrow f'(x) = 6x$$

2. Insert the x_0-value into $f(x)$ to obtain the y_0-value

$$\Rightarrow y_0 = 3 \cdot 1^2 + 1$$
$$y_0 = 4$$

3. Insert the x_0-value into $f'(x)$ to obtain m

$$\Rightarrow f'(1) = 6 \cdot 1$$
$$f'(1) = 6$$



4. Insert m and y into the general form of the equation of a straight line to obtain b

$$\Rightarrow y = m \cdot x + b$$
$$4 = 6 \cdot 1 + b$$
$$b = -2$$

5. Equation of the tangent

$$\Rightarrow y = 6x - 2$$

9.3.16 Normal Lines to a Curve

A normal line to a curve is perpendicular (orthogonal) to the corresponding tangent at the meeting point with the function $f(x)$. Its slope is equal to the negative reciprocal of the corresponding tangent.

$$\Rightarrow m_{norm} = -\frac{1}{m_{tan}} = -\frac{1}{f'(x_0)}$$

Example:

given: $f(x) = 3x^2 + 1$ $\quad x_0 = 1$

to be found: $y = m \cdot x + b$ $\quad$ (equation of normal line)

1. Determine the derivative of $f(x)$ and the slope of the tangent m_{tan}

$$\Rightarrow f'(1) \ = \ 6 \ = \ m_{tan}$$

2. Determine the slope of the normal line

$$\Rightarrow m_{norm} = \frac{-1}{m_{tan}} = \frac{-1}{6}$$

3. Insert m_{norm} and $P(1|4)$ into the form of the equation of a straight line to obtain b

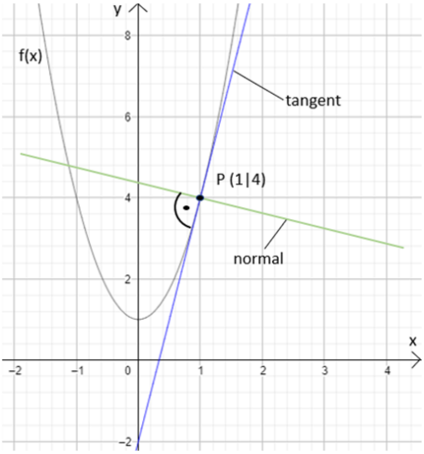
$$\Rightarrow 4 = -\frac{1}{6} \cdot 1 + b$$

$$b = \frac{25}{6}$$

4. Equation of the normal line

$$\Rightarrow y = -\frac{1}{6}x + \frac{25}{6}$$

Note:
$m_{norm} \cdot m_{tan} = -1$
must always apply.



9.4 Exercises

Example 1:

The water level of the Hudson river in NYC from the 6th to the 12th of January 2011 is modelled realistically by the function
$h(t) = 2.5t^2 \cdot e^{-\frac{1}{2}t} + 3.5.$

t is given in days, $t = 0$ corresponds to January 6th, $h(t)$ is measured in metres.

1. Determine the time when the flood as well as the water level reach their maximum.
2. Determine the times when the water level rises or falls the most.

Subtask 1:

Form the derivatives:

$$h'(t) = e^{-\frac{1}{2}t}(-1.25t^2 + 5t)$$

$$h''(t) = e^{-\frac{1}{2}t}(0.625t^2 - 5t + 5)$$

$$h'''(t) = e^{-\frac{1}{2}t}(-0.3125t^2 + 3.75t - 7.5)$$

1. Necessary condition $h'(t) = 0$

$$\begin{array}{llll}
 & e^{-\frac{1}{2}t}(-1.25t^2 + 5t) & = 0 & \mid e^{-\frac{1}{2}t} \neq 0 \text{ for all } t \in \mathbb{R} \\
 & -1.25t^2 + 5t & = 0 & \mid \text{factorise } t \\
t_1 = 0 \quad \vee & (-1.25t + 5) & = 0 & \mid -5 \\
 & -1.25t & = -5 & \mid \div(-1.25) \\
t_2 = 4 & & &
\end{array}$$

2. Sufficient condition $h'(t) = 0 \quad \wedge \quad h''(t) \neq 0$

$$\begin{array}{lll} h''(0) = 5 & > 0 & \text{local minimum} \\ h''(4) \approx -0.677 & < 0 & \text{local maximum} \end{array}$$

Determine the $y-$values

$$h(4) \approx 8.9$$

Answer: The water level of the Hudson river reached its peak of about 8.9 metres on January 10th at 12 a.m.

<u>Subtask 2:</u>

$h''(t) = e^{-\frac{1}{2}t}(0.625t^2 - 5t + 5)$

1. Necessary condition $h''(t) = 0$

$$\begin{array}{rlll} & e^{-\frac{1}{2}t}(0.625t^2 - 5t + 5) & = 0 & \mid e^{-\frac{1}{2}t} \neq 0 \text{ for all } t \in \mathbb{R} \\ & 0.625t^2 - 5t + 5 & = 0 & \mid : 0.625 \\ & t^2 - 8t + 8 & = 0 & \mid p/q \text{ formula} \\ t_{1,2} = & \frac{8}{2} \pm \sqrt{(-\frac{8}{2})^2 - 8} & & \mid t_1 \approx 6.828 \quad t_2 \approx 1.172 \end{array}$$

2. Sufficient condition $h''(t) = 0 \quad \wedge \quad h'''(t) \neq 0$

$h'''(t) = e^{-\frac{1}{2}t}(-0.3125t^2 + 3.75t - 7.5)$

$h'''(6.828) = 0.1163 > 0$ changing point from concave to convex

The derivative function h' has a local minimum at $t = 6.828$. The water level falls most sharply after about 6 days and 20 hours, i.e. on January 12th at around 8 p.m.

$h'''(1.172) \approx -1.968 < 0$ changing point from convex to concave

The derivative function h' has a local maximum at $t = 1.172$. The water level rises most significantly after about one day and 4 hours, i.e. on January 12th at around 4 a.m.

Example 2:

High ozone concentration can cause irritation of the respiratory tract, as well as coughing and lung diseases in human beings. Its extent is mainly determined by the duration spent in the contaminated air. According to the forecast for the following day, the ozone concentration in a German city between 7 a.m. ($t = 0$) and 9 p.m. ($t = 14$) is measured by the function f with the equation $f(t) = 0.06 \cdot (0.25t^4 - 10.6t^3 + 101.2t^2) + 55$ with $0 \leq t \leq 14$.

1. Determine the time when the highest ozone concentration in the city is predicted.

2. Determine the times when the ozone concentration in the city increases and decreases the most.

Subtask 1:

Form the derivatives:

$$f'(t) = 0.06 \cdot (t^3 - 31.8t^2 + 202.4t)$$

$$f''(t) = 0.06 \cdot (3t^2 - 63.6t + 202.4)$$

$$f'''(t) = 0.06 \cdot (6t - 63.6)$$

1. Necessary condition $f'(t) = 0$

$$\begin{array}{rlll} & 0.06 \cdot (t^3 - 31.8t^2 + 202.4t) & = 0 & \\ 0.06 \neq 0 \quad \vee & (t^3 - 31.8t^2 + 202.4t) & = 0 & \\ & (t^3 - 31.8t^2 + 202.4t) & = 0 & \mid \text{factorise } t \\ & t(t^2 - 31.8t + 202.4) & = 0 & \\ t_1 = 0 \quad \vee & (t^2 - 31.8t + 202.4) & = 0 & \mid p/q \text{ formula} \end{array}$$

$$\begin{array}{rl} & (t^2 - 31.8t + 202.4) = 0 \\ t_{2,3} = & 15.9 \pm \sqrt{(-15.9)^2 - 202.4} \\ t_2 = & 23 > 14; \quad t_3 \notin \mathbb{D} \\ t_3 = & 8.8 \end{array}$$

2. Sufficient condition $f'(t) = 0 \quad \wedge \quad f''(t) \neq 0$

$$\begin{array}{lll} f''(0) = 12.144 & > 0 & \text{local minimum} \\ f''(8.8) = -7.4976 & < 0 & \text{local maximum} \end{array}$$

At the time $t = 8.8$, the maximum is reached. This corresponds to 3:48 p.m. ($0.8h = 0.8 \cdot 60mins = 48mins$). The ozone concentration in the city reaches its peak at 3:48 p.m.

Subtask 2:

1. Necessary condition $f''(t) = 0$

$$\begin{array}{lllll} & & 0.06 \cdot (3t^2 - 63.6t + 202.4) & = 0 & \\ 0.06 \neq 0 & \vee & (3t^2 - 63.6t + 202.4) & = 0 & \\ & & (3t^2 - 63.6t + 202.4) & = 0 & |: 3 \\ & & t^2 - 21.2t + \dfrac{1,012}{15} & = 0 & \mid p/q \text{ formula} \end{array}$$

$$\begin{aligned} t_{1,2} = \quad & 10.6 \pm \sqrt{(-10.6)^2 - \frac{1,012}{15}} \\ & t_1 \approx 17.3 > 14 \notin \mathbb{D} \\ & t_2 \approx 3.9 \end{aligned}$$

2. Sufficient condition $f''(t) = 0 \quad \vee \quad f'''(t) \neq 0$

$f'''(3.9) = -2.412 \neq 0$ changing point from convex to concave

At the time $t = 3.9$, there is an inflection point.
The slope at this time: $f'(3.9) \approx 21.9$

Examine the slope in the neighbourhood of the time interval $[0;14]$:

$f'(0) = 0$
$f'(14) = -39.312$

The moment of the strongest increment is $t = 3.9$, i.e. at 10:54 a.m.
The moment of the strongest decrement is $t = 14$, i.e. at 9 p.m.

Chapter 10

Differential Calculus

10.1 Differentiation of Functions with One Independent Variable

10.1.1 General

Difference Quotient

= average slope of the function $f = f(x)$ between the points P_0 and P or between x_0 and x_1.

= the quotient (the relation) of the differences Δy and Δx (Fig. 10.1).

$$\frac{\Delta y}{\Delta x} = \frac{y - y_0}{x - x_0} = \frac{f(x_0 + \Delta x) - f(x_o)}{\Delta x}$$

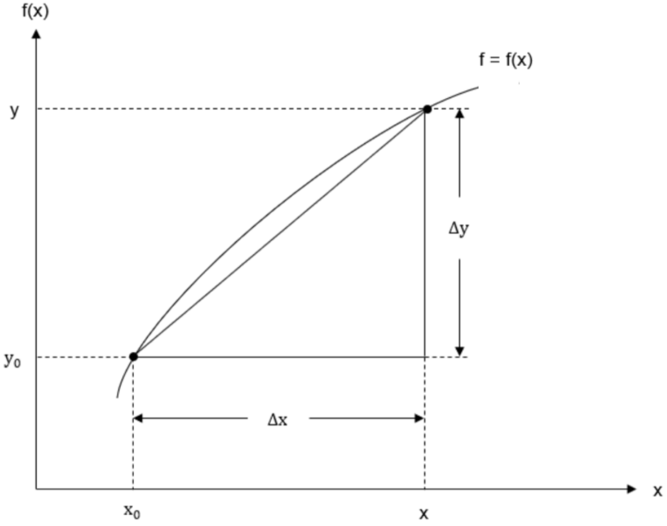


Fig. 10.1: The Difference Quotient of a Function $f = f(x)$

F. W. Peren, *Math for Business and Economics*,
https://doi.org/10.1007/978-3-662-68860-1_10

Differential Quotient

$=$ derivation (= slope) of the function $f = f(x)$ at the position x_0 or at/around the point P_0.

$$\begin{aligned}\frac{df}{dx}(x_0) &= f'(x_0)\\ &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}\end{aligned}$$

The progression from the difference quotient to the differential quotient is shown graphically in Fig. 10.2.

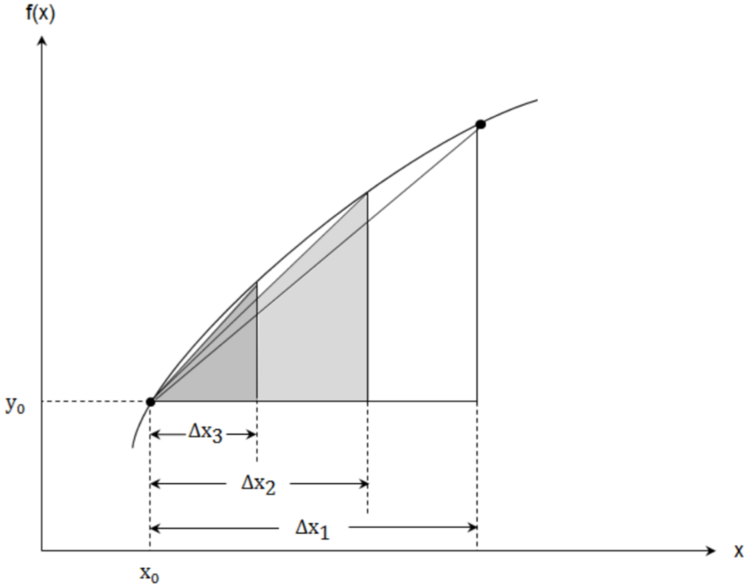


Fig. 10.2: Progression from the Difference Quotient to the Differential Quotient

Remark:

$\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ means geographically, that the slope triangle gets progressively smaller, theoretically so small that Δx converts to 0. Finally, the slope triangle at the point of interest P_0 is so small that it approximately measures the slope of the function $f = f(x)$ in/around P_0.

Derivative Function

If the function $f = f(x)$ is differentiable in the entire domain, the derivative function exists (1$^{\text{st}}$ derivative of f).

$$f'(x) = \frac{df}{dx} \qquad \text{with} \qquad x \in D_f$$

Differential

$$df = f'(x)dx \qquad \text{with} \qquad f = f(x);\, x \in D_f$$

df is called the differential of the function f.

10.1.2 First Derivative of Elementary Functions

$f(x)$	$f'(x)$	Remarks
c	0	$c =$ constant
x^n	$n \cdot x^{n-1}$	$n \in \mathbb{R}$
$c \cdot x^n$	$c \cdot n \cdot x^{n-1}$	
$\frac{1}{x} = x^{-1}$	$-\frac{1}{x^2} = -x^{-2}$	
$\frac{1}{x^n} = x^{-n}$	$-\frac{n}{x^{n+1}} = (-n) \cdot x^{-n-1}$	
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	
$\sqrt[n]{x}$	$\frac{1}{n} \cdot x^{\frac{1}{n}-1}$	
$\sqrt{g(x)}$	$\Rightarrow$ chain rule	
$\ln(x)$	$\frac{1}{x}$	
$\ln(g(x))$	$\Rightarrow$ chain rule	
$\log_a x$	$\frac{1}{x \cdot \ln(a)}$	$a \neq 1 \quad a, x > 0$
e^x	e^x	
c^x	$c^x \ln(c)$	$c > 0$
$e^{g(x)}$	$\Rightarrow$ chain rule	
$\sin x$	$\cos x$	

$f(x)$	$f'(x)$	Remarks
$\cos x$	$-\sin x$	
$\cot x$	$-\frac{1}{\sin^2 x} = -(1+\cot^2 x)$	$x \neq k\pi,\ k \in \mathbb{R}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\lvert x\rvert < 1$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$\lvert x\rvert < 1$
$\arctan x$	$\frac{1}{\sqrt{1+x^2}}$	
$\operatorname{arccot} x$	$-\frac{1}{\sqrt{1+x^2}}$	
$\sinh x$	$\cosh x$	
$\cosh x$	$\sinh x$	
$\tanh x$	$\frac{1}{\cosh^2 x} = 1-\tanh^2 x$	
$\coth x$	$-\frac{1}{\sinh^2 x} = 1-\coth^2 x$	$x \neq 0$
$\operatorname{arsinh} x$	$\frac{1}{\sqrt{1+x^2}}$	
$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$	$x > 1$
$\operatorname{artanh} x$	$\frac{1}{\sqrt{1-x^2}}$	$\lvert x\rvert < 1$
$\operatorname{arcoth} x$	$-\frac{1}{\sqrt{x^2-1}}$	$\lvert x\rvert > 1$

10.1.3 Derivation Rules

Constant Factor Rule $f(x) = c \cdot g(x)$ with $c =$ constant $c \in \mathbb{R}$

$$f'(x) = c \cdot g'(x)$$

Sum Rule $f(x) = u(x) \pm v(x)$

$$f'(x) = u'(x) \pm v'(x)$$

Product Rule $f(x) = u(x) \cdot v(x)$

$$f'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

In general:

$$f(x) = f_1(x) \cdot f_2(x) \cdot \ldots \cdot f_n(x)$$

$$f'(x) = f_1'(x) \cdot f_2(x) \cdot \ldots \cdot f_n(x) + f_1(x) \cdot f_2'(x) \cdot \ldots \cdot$$

$$\cdot f_n(x) + f_1(x) \cdot f_2(x) \cdot \ldots \cdot f_n'(x)$$

Quotient Rule $f(x) = \dfrac{u(x)}{v(x)}$ with $v(x) \neq 0$

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2}$$

Chain Rule $f(x) = u \cdot v = u(v(x))$ with $x \in D_v \quad C_v \subset D_u$

$$f'(x) = u'(v(x)) \cdot v'(x)$$

= outer times inner derivative

In general:

$$f(g_1(g_2(g_3 \ldots g_n(x))))$$

$$\frac{df}{dx} = \frac{df}{dg_1} \cdot \frac{dg_1}{dg_2} \cdot \frac{dg_2}{dg_3} \cdot \ldots \cdot \frac{dg_n}{dx}$$

<u>Examples:</u>

Constant Factor Rule: $f(x) = 5x^{20}$

$$\Rightarrow f'(x) = 5 \cdot 20x^{20-1}$$

$$\Rightarrow f'(x) = 100x^{19}$$

Sum Rule: $f(x) = 2e^x + 4\ln x - \frac{2}{\sqrt{x}}$

$$\Rightarrow f'(x) = 2e^x + \frac{4}{x} + \frac{1}{\sqrt{x^3}}$$

Product Rule: $f(x) = x^7 \cdot \ln x$

$$\Rightarrow f'(x) = 7x^6 \cdot \ln x + x^7 \cdot \frac{1}{x}$$

Quotient Rule: $f(x) = \frac{e^x}{x^4+1}$

$$\Rightarrow f'(x) = \frac{e^x \cdot (x^4+1) - e^x \cdot (4x^3)}{(x^4+1)^2}$$

Chain Rule:

$$f(x) = \left[\ln \frac{1}{x}\right]^{0.5}$$

$$\Rightarrow f'(x) = 0.5 \cdot \ln\left(\frac{1}{x}\right)^{-0.5} \cdot \frac{1}{\frac{1}{x}} \cdot (-1) \cdot x^2$$

10.1.4 Higher Derivations

gen. (recursive) definition

$$f^{(n+1)}(x) = \left(f^{(n)}(x)\right)' = \frac{df^{(n)}(x)}{dx}$$

with $n \in \mathbb{Z}^*$

2nd derivative

$$f''(x) = \left(f'(x)\right)' = \frac{d^2 f(x)}{dx^2}$$

3rd derivative

$$f'''(x) = \left(f''(x)\right)' = \frac{d^3 f(x)}{dx^3}$$

4th derivative

$$f^{(4)}(x) = \left(f'''(x)\right)' = \frac{d^4 f(x)}{dx^4}$$

nth derivative

$$f^{(n)}(x) = \left(f^{(n-1)}(x)\right)' = \frac{d^n f(x)}{dx^n}$$

<u>Examples:</u>

(1)

$$f(x) = \frac{1}{4}x^4 - \frac{1}{6}x^3 + 2x + 1$$

$$f'(x) = x^3 - \frac{1}{2}x^2 + 2$$

$$f''(x) = 3x^2 - 1x$$

$$f'''(x) = 6x - 1$$

(2)

$$f(x) = lnx$$

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f'''(x) = \frac{1 \cdot 2}{x^3}$$

$$f^{(4)}(x) = 6 \qquad f^{(4)}(x) = -\frac{1 \cdot 2 \cdot 3}{x^4}$$

$$f^{(5)}(x) = 0 \qquad f^{(5)}(x) = \frac{1 \cdot 2 \cdot 3 \cdot 4}{x^5}$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(x) = 0 \qquad f^{(n)}(x) = (-1)^{n-1} \cdot \frac{(n-1)!}{x^n}$$

10.1.5 Differentiation of Functions with Parameters

$f = f(x)$ indicates that x is the (only) independent variable of the function f. If other placeholders appear in the function term, these are parameters, not variables. Accordingly, f cannot be differentiated with respect to them.

Example:

$$f(x) = 2x^2z - z^3 + z^2 lnx$$

$$f'(x) = 4xz - z^2\frac{1}{x}$$

$f'(z)$ not possible, as z is a parameter.

10.1.6 Curve Sketching

An analysis of the function $f = f(x)$ generally includes the complete examination of the following criteria:

Domain D_f

including discontinuities; check for possible removable discontinuities.

Codomain C_f

is usually recommended as the last point of sketching.

Symmetry

$f(x) = f(-x)$ f is axially symmetric to the y-axis

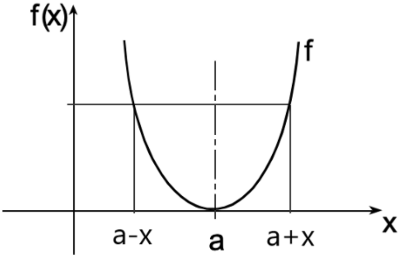
$f(x) = -f(-x)$ f is point symmetric to the point of origin

Axial Symmetry

If the function f, with $f = f(x)$, is axially symmetric to the axis of symmetry $x = a$, the following applies to all $x \in D_f$:

$f(a+x) = f(a-x)$

with $a =$ constant
(= axis of symmetry)



If $a = 0$, a is equivalent to the y-axis.

Then the following is valid: $f(x) = f(-x)$

A polynomial function f, whose graph is axially symmetric to the y-axis, has the form

$f(x) = q_n x^n + q_{n-2} x^{n-2} + \ldots + q_2 x^2 + q_0$

with $q =$ constant and $n =$ even.

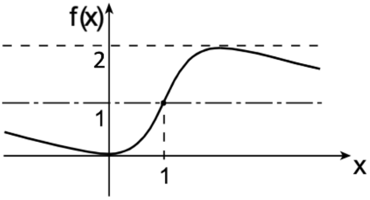
Point Symmetry

If the function f, with $f = f(x)$, is point symmetric to an arbitrary point $P = P(a,b)$, the following applies for all $x \in D_f$:

$f(a+x) - f(a) = -f(a-x) + f(a)$

with $a =$ const. $= x$-coordinate of P

with $b =$ const. $= y$-coordinate of P



Zeros = Intercepts with the x-Axis

$f(x) \overset{!}{=} 0$

Intercepts with the y-Axis

$x \overset{!}{=} 0$ (periodicity)

$f(x) = f(x \pm n \cdot T)$ $T =$ period of f; $n \in \mathbb{Z}^*$; $T > 0$

Continuity/Differentiability

If the function f, with $f = f(x)$, can be differentiated in D_f (or in intervals of D_f), it is also continuous here.

Incl. discontinuities (poles, gaps, jumps).

Pole at $x = x_P$	$\lim\limits_{x \to x_P^-} f(x)$; $\lim\limits_{x \to x_P^+} f(x)$
Removable discontinuity at $x = x_{Dis}$	These discontinuities are removable by definition; with $D_f = \mathbb{R} \setminus \{x_{Dis}\}$
Jump at $x = x_J$	$\lim\limits_{x \to x_J^-} f(x) \neq \lim\limits_{x \to x_J^+} f(x)$ Clearly allocate limits of intervals

Extrema

f has a relative/local maximum at position x_0, if the following is valid:

$f'(x_0) = 0$	necessary condition
$f''(x_0) < 0$	sufficient condition

f has a relative/local minimum at position x_0, if the following is valid:

$f'(x_0) = 0$	necessary condition
$f''(x_0) > 0$	sufficient condition

In the extrema the function changes its monotonic behaviour.

Inflection Points

f has an inflection point at position x_0, if the following is valid:

$f''(x_0) = 0$	necessary condition
$f'''(x_0) \neq 0$	sufficient condition

In the inflection points the function changes its curvature behaviour:

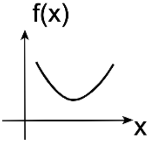
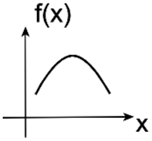
$f'''(x_0) < 0$	convex/concave inflection point
$f'''(x_0) > 0$	concave/convex inflection point

Furthermore, if $f'(x_0) = 0$, then this is a special kind of inflection point, namely a saddle point. The saddle point is an inflection point with a slope of zero and, therefore, it has a horizontal tangent.

Monotonicity

$f'(x_0) \geq 0$	f (in the interval) monotonically increasing
$f'(x_0) > 0$	f (in the interval) strictly monotonically increasing
$f'(x_0) \leq 0$	f (in the interval) monotonically decreasing
$f'(x_0) < 0$	f (in the interval) strictly monotonically decreasing

Curvature

$f''(x_0) > 0$	f (in the interval) convex (curvature left)	
$f''(x_0) < 0$	f (in the interval) concave (curvature right)	

Curvature radius: $p = \dfrac{\sqrt{(1+(f'(x))^2)^3}}{f''(x)}$ with $f''(x) \neq 0$

Centre $C(x_C; y_C)$ of the osculating circle with the coordinates:

$$x_C = x - \frac{f'(x) \cdot (1+(f'(x))^2)}{f''(x)}$$

$$y_C = f(x_C) = f(x) + \frac{(1+(f'(x))^2)}{f''(x)}$$

Behaviour at the Boundaries

$\lim\limits_{x \to -\infty} f(x)$	left boundary of f
$\lim\limits_{x \to +\infty} f(x)$	right boundary of f

The respective optimum optimorum (smallest minimum/ largest maximum) can lie in a relative/ local extremum as well as at one or both boundaries.

Asymptotes

An asymptote is a function to which another function increasingly converges.

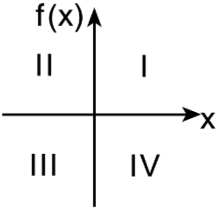
Asymptotes of broken rational functions:

$n =$ degree of numerator $m =$ degree of denominator

$n < m$	x-axis is horizontal asymptote
$n = m$	line parallel to the x-axis is an asymptote
$n = m + 1$	skewed asymptote
$n > m + 1$	curved asymptote
Numerator $= 0$	perpendicular x-axis (at discontinuities/ poles)
	even multiplicity of the zero
	$\Rightarrow$ with change of sign
	odd multiplicity of the zero
	$\Rightarrow$ without change of sign

Graph of the Function

The graphic plane $f = f(x)$ is spanned by a two-dimensional Cartesian coordinate system.



Example of Curve Sketching:

$$f(x) = \frac{5x-4}{(2-3x)^2}$$

Domain: $D_f = \mathbb{R}\backslash\left\{\frac{2}{3}\right\}$

Codomain: $C_f =]-\infty\,;\ \frac{25}{24}\,]$

Symmetry:

(a) Axial symmetry to the y-axis $f(x) = f(-x)$

$$\frac{5x-4}{(2-3x)^2} \neq \frac{5(-x)-4}{(2-3(-x))^2}$$

(b) Point symmetry to the origin $f(x) = -f(-x)$

$$\frac{5x-4}{(2-3x)^2} \neq -\left(\frac{5(-x)-4}{(2-3(-x))^2}\right)$$

$\Rightarrow f$ is neither axially nor point symmetric (also seen in graph of function).

Zeros:

$$f(x) \overset{!}{=} 0$$

$$5x-4 \overset{!}{=} 0$$

$$x = \frac{4}{5}$$

$\Rightarrow f$ has a zero at point $Z_1\left(\frac{4}{5}\,|\,0\right)$

Intercepts with the y-Axis: $x \overset{!}{=} 0$

$$\frac{5\cdot 0-4}{(2-3\cdot 0)^2}=\frac{-4}{4}=-1$$

Periodicity: $f(x)\neq f(x\pm n\cdot T)$ with $n\in\mathbb{Z}^*$ $T>0$

$\Rightarrow f$ is not periodic

Continuity: f is continuous in $D_f=\mathbb{R}\backslash\left\{\frac{2}{3}\right\}$ as differentiable.

$\Rightarrow f$ is discontinuous in $x=\dfrac{2}{3}$

$\Rightarrow$ pole

$$\lim_{x\to\frac{2}{3}^+} f(x)\approx f(0.6667)=-\infty$$

$$\lim_{x\to\frac{2}{3}^-} f(x)\approx f(0.6665)=-\infty$$

Extrema: $f(x)=\dfrac{5x-4}{(2-3x)^2}=\dfrac{u(x)}{v(x)}$

$$f'(x)=\frac{u'(x)v(x)-u(x)v'(x)}{(v(x))^2}$$

$$f'(x)=\frac{5\cdot(2-3x)^2-((5x-4)\cdot 2(2-3x)\cdot(-3))}{(2-3x)^4}$$

$$=\frac{5\cdot(2-3x)-((5x-4)\cdot 2\cdot(-3))}{(2-3x)^3}$$

$$=\frac{10-15x+30x-24}{(2-3x)^3}=\frac{15x-14}{(2-3x)^3}$$

$$\Rightarrow\frac{15x-14}{(2-3x)^3}\overset{!}{=}0\Rightarrow 15x-14\overset{!}{=}0\Rightarrow x=\frac{14}{15}$$

$$
\begin{aligned}
f''(x) &= \frac{15\cdot(2-3x)^3-((15x-14)\cdot 3(2-3x)^2\cdot(-3))}{(2-3x)^6} \\
&= \frac{15\cdot(2-3x)-((15x-14)\cdot 3\cdot(-3))}{(2-3x)^4} \\
&= \frac{30-45x-(-135x+126)}{(2-3x)^4} \\
&= \frac{90x-96}{(2-3x)^4}
\end{aligned}
$$

$$f''\left(\tfrac{14}{15}\right) = -\frac{1875}{64} < 0 \Rightarrow \text{maximum at } x = \frac{14}{15}$$

$$f\left(\tfrac{14}{15}\right) = 0.452 \Rightarrow \text{maximum at } E_{max}\left(\tfrac{14}{15} \mid \tfrac{25}{24}\right)$$

Inflection Points: $f''(x) = \dfrac{90x-96}{(2-3x)^4} \overset{!}{=} 0 \quad \Rightarrow 90x-96 \overset{!}{=} 0 \quad \Rightarrow x = \dfrac{16}{15}$

$$f'''(x) = \frac{810x-972}{(2-3x)^5}$$

$$f'''\left(\tfrac{16}{15}\right) = \frac{-108}{(-\frac{6}{5})^5} \approx 43.403 > 0$$

$\Rightarrow$ concave/convex inflection point at $IP_1\left(\tfrac{16}{15} \mid \tfrac{25}{27}\right)$

Monotonicity:

$]-\infty\,;\,\frac{2}{3}[$	f decreases strictly monotonically
$]\frac{2}{3}\,;\,\frac{14}{15}[$	f increases strictly monotonically
$]\frac{14}{15}\,;\,+\infty[$	f decreases strictly monotonically

Curvature:

$]-\infty\,;\,\frac{2}{3}[$	concave
$]\frac{2}{3}\,;\,\frac{16}{15}[$	concave
$]\frac{16}{15}\,;\,+\infty[$	convex

Behaviour at the Boundaries:

$$\lim_{x\to+\infty} f(x) \approx f(1000) = 0^+$$

$$\lim_{x\to-\infty} f(x) \approx f(-1000) = 0^-$$

Asymptotes:

$$f(x) = \frac{5x-4}{(2-3x)^2} \Rightarrow \frac{n=1}{m=2} \Rightarrow n < m$$

$\Rightarrow$ x-axis (abscissa) is horizontal asymptote

denominator $= 0$

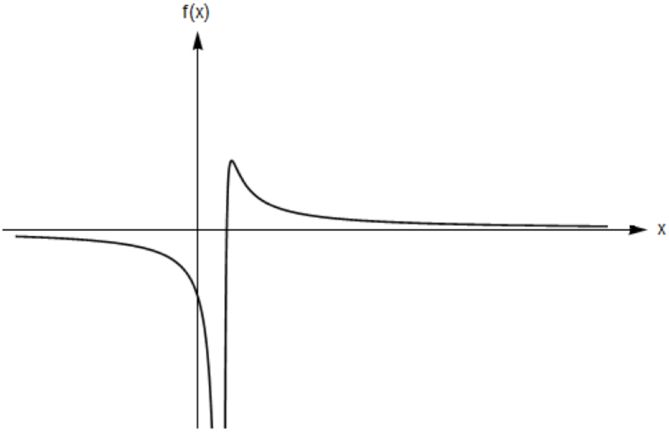
$$(2-3x)^2 = (2-3x)\cdot(2-3x) = 0$$

$$\Rightarrow 2-3x = 0 \Rightarrow x = \frac{2}{3}$$

$\Rightarrow$ perpendicular asymptote at $x = \frac{2}{3}$

no change of signs because of even multiplicity

Graph of Function:



10.2 Differentiation of Functions with More Than One Independent Variable

$$f = f(x_1,\, x_2, \ldots,\, x_k, \ldots,\, x_n)$$

10.2.1 Partial Derivatives (1st Order)

Partial differential operator $\dfrac{\partial f}{\partial x_k}$ (read "f partially derived with respect to x_k")

with

$$\frac{\partial f}{\partial x_k} = f'_{x_k} = \lim_{\Delta x_k \to 0} \frac{f(x_1,\, \ldots,\, x_k + \Delta x_k,\, \ldots,\, x_n) - f(x_1,\, \ldots,\, x_n)}{\Delta x_k}$$

$$k = 1, 2, \ldots, n$$

Geometrically, $\dfrac{\partial f}{\partial x_k}$ is equivalent to the slope function of f in the direction of the x_k-axis. If the slope of f at point P_0 with the coordinates $(x_{10},\, x_{20},\, \ldots,\, x_{k0},\, \ldots,\, x_{n0})$ in the direction of x_k is of interest, the function f is to be partially derived with respect to x_k, $\dfrac{\partial f}{\partial x_k}$. The coordinates of P_0 are to be inserted within this slope function. With the partial derivation of f with respect to x_k, all other variables are regareded as constants, ceteris paribus.

If $f(x,y)$ is differentiable in $P_0(x_0,y_0)$, the following is valid:

$$\frac{\partial f(x_0,\, y_0)}{\partial x} = \left.\frac{\partial f}{\partial x}\right|_{(x=x_0,\, y_0)} = f'_x(x_0,\, y_0)\, \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x,\, y_0) - f(x_0,\, y_0)}{\Delta x}$$

$$\frac{\partial f(x_0,\, y_0)}{\partial y} = \left.\frac{\partial f}{\partial y}\right|_{(x_0,\, y=y_0)} = f'_y(x_0,\, y_0)\, \lim_{\Delta y \to 0} \frac{f(x_0,\, y_0 + \Delta y) - f(x_0,\, y_0)}{\Delta y}$$

Examples:

(1) $f(x,y) = x^2 - 6x + 2xy^2 - 7 \Rightarrow$ for point $P_1\ (7\,|\,2)$

$\frac{\partial f}{\partial x} \,\widehat{=}\,$ slope of f in x-direction $= 2x - 6 + 2y^2$

$f'x\,(7\,|\,2) = 2 \cdot 7 - 6 + 2 \cdot 2^2 = 16$

$\frac{\partial f}{\partial y} \,\widehat{=}\,$ slope of f in y-direction $= 4xy$

$f'y\,(7\,|\,2) = 4 \cdot 7 \cdot 2 = 56$

(2) $f(x;y;z) = x \cdot e^{yz} + \frac{\sqrt{x \cdot z}}{lny} \Rightarrow$ for point $P_2\ (1\,|\,2\,|\,1)$

$$\frac{\partial f}{\partial x} \,\widehat{=}\, e^{yz} + \frac{1}{lny} \cdot \frac{1}{2\sqrt{xz}} \cdot z$$

$$f'x(1\,|\,2\,|\,1) = e^{2 \cdot 1} + \frac{1}{ln2} \cdot \frac{1}{2\sqrt{1 \cdot 1}} \cdot 1 = 8.1104$$

$$\frac{\partial f}{\partial y} \,\widehat{=}\, x \cdot e^{yz} \cdot z + \sqrt{xz} \cdot (-1) \cdot (lny)^{-2} \cdot \frac{1}{y}$$

$$f'y(1\,|\,2\,|\,1) = 1 \cdot e^{2 \cdot 1} \cdot 1 + \sqrt{1 \cdot 1} \cdot (-1) \cdot (ln2)^{-2} \cdot \frac{1}{2} = 6.3484$$

$$\frac{\partial f}{\partial z} \,\widehat{=}\, x \cdot e^{yz} \cdot y + \frac{1}{lny} \cdot \frac{1}{2\sqrt{xz}} \cdot x$$

$$f'z(1\,|\,2\,|\,1) = 1 \cdot e^{2 \cdot 1} \cdot 2 + \frac{1}{ln2} \cdot \frac{1}{2\sqrt{1 \cdot 1}} \cdot 1 = 15.4995$$

Geometrical Interpretation

(1) The slope of the function $f = f(x,y)$ at point $P_1\,(x=7\,;\; y=2)$ is 16 units in the direction of the x-axis and 56 units in direction of the y-axis.

(2) The slope of the function $f = f(x,y,z)$ at point $P_2\,(x=1;\; y=2;\; z=1)$ is 8.1104 units in the direction of the x-axis, 6.3484 units in the direction of the y-axis and 15.4995 units in the direction of the z-axis.

Economic Interpretation

(1) If, starting from the status quo in P_1, the x-value is increased (decreased) ceteris paribus (c.p.) by one unit, the function value f increases (decreases) by approximately 16 units. If the y-value is increased (decreased) c.p. by one unit, f increases (decreases) by approximately 56 units.

(2) If, starting from the status quo in P_2, the x-value is increased (decreased) c.p. by one unit, the function value f increases (decreases) by approximately 8.1104 units. If the y-value increases (decreases) c.p. by one unit, f increases (decreases) by approximately 6.3484 units. If the z-value is increased (decreased) c.p. by one unit, f increases (decreases) by approximately 15.4995 units.

10.2.2 Partial Derivatives (2$^{\text{nd}}$ Order)

$$\frac{\partial^2 f}{\partial x^2} = f''_{xx} \quad ; \quad \frac{\partial^2 f}{\partial y^2} = f''_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = f''_{xy} \quad ; \quad \frac{\partial^2 f}{\partial y \partial x} = f''_{yx} \quad \Rightarrow \quad \text{mixed partial derivative}$$

Schwarz' Theorem[1]

Provided that the function $f = f(x,y)$ is partially differentiable with respect to x and y and that the two mixed partial derivatives (in the relevant interval) are continuous, the following applies:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \qquad \text{equality of the mixed partial derivatives}$$

or $f''_{xy} = f''_{yx}$

<u>Example:</u> $f(x;y) = 2x^4y^3 - x^3y^6$

$$\frac{\partial f}{\partial x} = f'_x = 8x^3y^3 - 3x^2y^6$$

$$\frac{\partial f}{\partial y} = f'_y = 6x^4y^2 - 6x^3y^5$$

$$\frac{\partial^2 f}{\partial x^2} = f''_{xx} = 24x^2y^3 - 6xy^6$$

---

[1] Hermann Amandus Schwarz (1843 - 1921) was a German mathematician and university lecturer in Berlin.

$$\frac{\partial^2 f}{\partial y^2} = f''_{yy} = 12x^4y - 30x^3y^4$$

$$\frac{\partial^2 f}{\partial x \partial y} = f''_{xy} = 24x^3y^2 - 18x^2y^5$$

$$\frac{\partial^2 f}{\partial y \partial x} = f''_{yx} = 24x^3y^2 - 18x^2y^5$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = f'''_{yyx} = 48x^3y - 90x^2y^4$$

$$\frac{\partial^4 f}{\partial y^3 \partial x} = f^{(4)}_{yyyx} = 48x^3 - 360x^2y^3$$

Partial Derivative of r^{th} Order

If the function f, with $f = f(x_1, \ldots, x_n)$, is partially differentiable r times, with $r \geq 2$ ($r =$ order of partial derivatives; number of differentiations), the order of the partial derivatives can be arbitrarily interchanged.

Example: For $f = f(x, y)$ the following 3rd order

partial derivatives generally apply:

Number of 3rd order partial derivatives:

$$f'''_{xxx},\ f'''_{xxy},\ f'''_{xyx},\ f'''_{yxx},\ f'''_{xyy},\ f'''_{yxy},\ f'''_{yyx},\ f'''_{yyy}$$

where $f'''_{xxy} = f'''_{xyx} = f'''_{yxx}$

and $f'''_{yyx} = f'''_{yxy} = f'''_{xyy}$

10.2.3 Local Extrema of the Function $f = f(x, y)$

= Identification of relative maxima and minima of a function $f = f(x, y)$ with two independent variables, x and y, in three-dimensional space.

10.2.3.1 Relative Extrema without Constraint of the Function $f = f(x, y)$

A local relative extremum of a function $f = f(x, y)$ with two variables $x, y \in \mathbb{R}$ at a point $P_0(x_0, y_0)$ with $D_f \subset \mathbb{R}^2$ exists if the following applies:

necessary conditions

$$\frac{\partial f}{\partial x}(x_0, y_0) = f'_x(x_0, y_0) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x_0, y_0) = f'_y(x_0, y_0) = 0$$

The *tangent plane* to P_0 is parallel to the (x,y)-plane.

sufficient conditions

The sufficient conditions are derived from the so-called Hessian matrix . The Hessian matrix arranges all second partial derivatives of a function $f = f(x_1, ..., x_n)$ in a way defined by *Hesse*[2] and *Jacobi*[3]. It describes the local curvature behaviour of the function $f = f(x_1, ..., x_n)$ with several variables, $x_i \in \mathbb{R}$ and $i = 1, ..., n$, around a point $P = (x_1, ..., x_n) \in D_f$ with $D_f \subset \mathbb{R}^n$.

If the function $f = f(x_1, ..., x_n)$ is a twice continuously differentiable function the Hessian matrix of $f = f(x_1, ..., x_n)$ at a point $x = (x_1, ..., x_n) \in D_f$ with $D_f \subset \mathbb{R}^n$, $H_f(x)$, in general is defined as follows:

$$H_f(x) := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right)_{i,\, j=1,\, ...,\, n} =$$

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[2] Ludwig Otto Hesse (1811 - 1874) was a German mathematician.

[3] Carl Gustav Jacobi, originally Jacques Simon Jacobi, (1804 - 1851) was a German mathematician.

$$= \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(x) \end{pmatrix}$$

For a function $f = f(x, y)$ with two independent variables, x and y, in a three-dimensional space, the Hessian matrix, $H_f(x, y)$, at a point $(x_0, y_0) \in D_f$ with $D_f \subset \mathbb{R}^2$, is defined as follows:

$$H_f(x_0, y_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial^2 x}(x_0, y_0) & \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) \\ \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) & \frac{\partial^2 f}{\partial^2 y}(x_0, y_0) \end{bmatrix} = \begin{pmatrix} f''_{xx}(x_0, y_0) & f''_{xy}(x_0, y_0) \\ f''_{yx}(x_0, y_0) & f''_{yy}(x_0, y_0) \end{pmatrix}$$

The determinant of the Hessian matrix, $H_f(x)$,

det $H_f(x_0, y_0) = f''_{xx}(x_0, y_0) \cdot f''_{yy}(x_0, y_0) - (f''_{xy}(x_0, y_0))^2$,

describes the curvature behavior of the function $f = f(x, y)$ at the coordinate (x_0, y_0) as follows:

- det $H_f(x_0,\, y_0) > 0 \;\Rightarrow$ extremum at point $P_0(x_0,\, y_0,\, z_0)$ with $z_0 = f(x_0,\, y_0)$

  if $f''_{xx}(x_0,\, y_0) < 0$ and $f''_{yy}(x_0,\, y_0) < 0 \;\Rightarrow$ maximum at point $P_0(x_0,\, y_0,\, z_0)$

  if $f''_{xx}(x_0,\, y_0) > 0$ and $f''_{yy}(x_0,\, y_0) > 0 \;\Rightarrow$ minimum at point $P_0(x_0,\, y_0,\, z_0)$

- det $H_f(x_0,\, y_0) < 0 \;\Rightarrow$ saddle point at point $P_0(x_0,\, y_0,\, z_0)$ with $z = f(x_0,\, y_0)$

  with $f''_{xx}(x_0,\, y_0) > 0$ and $f''_{yy}(x_0,\, y_0) < 0 \Rightarrow$ saddle point, convex in x direction and concave in y direction

  or $f''_{xx}(x_0,\, y_0) < 0$ and $f''_{yy}(x_0,\, y_0) > 0 \Rightarrow$ saddle point, concave in x direction and convex in y direction

- det $H_f(x_0,\, y_0) = 0 \Rightarrow$ Indifference, i.e. a decision whether a relative extremum or a saddle point is present at $P_0(x_0,\, y_0,\, z_0)$ is not possible. In this case, either the curvatures in x and y directions are to be measured separately or the function value at P_0 with $z_0 = f(x_0,\, y_0)$, must be compared to neighbouring values.

In other notation, the procedure can be described as follows:

- $f''_{xx}(x_0,\, y_0) \cdot f''_{yy}(x_0,\, y_0) > (f''_{xy}(x_0,\, y_0))^2$

  $\Rightarrow$ extremum at point $P_0(x_0,\, y_0,\, z_0)$ with $z = f(x_0,\, y_0)$

  if $f''_{xx}(x_0,\, y_0) < 0$ and $f''_{yy}(x_0,\, y_0) < 0 \;\Rightarrow$ maximum at point $P_0(x_0,\, y_0,\, z_0)$

if $f''_{xx}(x_0,\, y_0) > 0$ and $f''_{yy}(x_0,\, y_0) > 0 \;\Rightarrow$ minimum at point $P_0(x_0,\, y_0,\, z_0)$

- $f''_{xx}(x_0,\, y_0) \cdot f''_{yy}(x_0,\, y_0) < (f''_{xy}(x_0,\, y_0))^2$

  $\Rightarrow$ saddle point at point $P_0(x_0,\, y_0,\, z_0)$ with $z_0 = f(x_0,\, y_0)$

  with $f''_{xx}(x_0,\, y_0) > 0$ and $f''_{yy}(x_0,\, y_0) < 0 \;\Rightarrow$ saddle point, convex in x direction and concave in y direction

  or $f''_{xx}(x,\, y) < 0$ and $f''_{yy}(x,\, y) > 0 \;\Rightarrow$ saddle point, concave in x direction and convex in y direction

- $f''_{xx}(x_0,\, y_0) \cdot f''_{yy}(x_0,\, y_0) = (f''_{xy}(x_0,\, y_0))^2$

  $\Rightarrow$ Indifference, i.e. a decision whether a relative extremum or a saddle point is present at $P_0(x_0,\, y_0,\, z_0)$ is not possible. In this case, either the curvatures in x and y directions are to be measured separately or the function value at P_0 with $z_0 = f(x_0,\, y_0)$, must be compared to neighbouring values.

Example 1: $f(x;y) = x^3 + 3x^2y - 3xy^2 - 21x + y^3 - 3y$

1st step:

$$f'(x) = 3x^2 + 6xy - 3y^2 - 21$$
$$f'_y = 3x^2 - 6xy + 3y^2 - 3$$
$$f''_{xx} = 6x + 6y$$
$$f''_{yy} = -6x + 6y$$
$$f''_{xy} = 6x - 6y$$

equation I $f'_x(x,\, y) = 3x^2 + 6xy - 3y^2 - 21 \overset{!}{=} 0$

equation II $f'_y(x,\, y) = 3x^2 - 6xy + 3y^2 - 3 \overset{!}{=} 0$

Addition method, since all $y's$ cancel out.

$$\begin{array}{rl} \text{I} & 3x^2+6xy-3y^2-21 \\ \text{II} & 3x^2-6xy+3y^2-3 \quad + \\ \hline & 6x^2 \qquad\qquad -24=0 \\ & \qquad\qquad 6x^2=24 \\ & \qquad\qquad x^2=4 \end{array}$$

$$\Rightarrow x_1 = 2 \text{ and } x_2 = -2$$

Insert the values for x_1 and x_2 in one of the first order derivatives ($f'x$ or $f'y$) to determine the corresponding y-values.

(1) for $x_0 = 2$: (insert in I)

$$0 = 3\cdot 2^2+6\cdot 2y-3y^2-21$$

$$0 = 12+12y-3y^2-21$$

$$0 = -3y^2+12y-9 \mid : (-3)$$

$$0 = y^2-4y+3 \qquad \Rightarrow 2\pm\sqrt{4-3}$$

$$2\pm 1$$

$$y_{11} = 3 \text{ and } y_{12} = 1$$

$$\Rightarrow P_1(2\,|\,3);\;\; P_2(2\,|\,1)$$

(2) for $x_0 = -2$: (insert in I)

$$0 = 3\cdot(-2)^2+6\cdot(-2)y-3y^2-21$$

$$0 = 12-12y-3y^2-21$$

$$0 = -3y^2-12y-9| : (-3)$$

$$0 = y^2+4y+3 \qquad \Rightarrow -2\pm\sqrt{4-3}$$

$$-2\pm 1$$

$$y_{21} = -3 \text{ and } y_{22} = -1$$

$$\Rightarrow P_3(-2\,|-1);\;\; P_4(-2\,|-3)$$

2nd step:

$P_1(2\,|\,3)$

$$\Rightarrow \underbrace{(6\cdot 2+6\cdot 3)}_{f''_{xx}(2\,|\,3)}\cdot\underbrace{(-6\cdot 2+6\cdot 3)}_{f''_{yy}(2\,|\,3)} \gtreqless \underbrace{(6\cdot 2-6\cdot 3)^2}_{f''_{xy}(2\,|\,3)}$$

$$30 \quad\cdot\quad 6 \quad > \quad 36$$

$180 > 36$ $\Rightarrow$ extremum at P_1

$\Rightarrow 30 > 0\ ;\ 6 > 0$ $\Rightarrow$ minimum at P_1

$P_2(2\,|\,1)$

$\Rightarrow (6\cdot 2+6\cdot 1)\cdot(-6\cdot 2+6\cdot 1) < (6\cdot 2-6\cdot 1)^2$

$-108 < 36$ $\Rightarrow$ saddle point at P_2

$P_3(-2\,|\,-1)$

$\Rightarrow [6\cdot(-2)+6\cdot(-1)]\cdot[(-6)\cdot(-2)+6\cdot(-1)] < [6\cdot(-2)-6\cdot(-1)]^2$

$-108 < 36$ $\Rightarrow$ saddle point at P_3

$P_4(-2|-3)$

$\Rightarrow [6\cdot(-2)+6\cdot(-3)]\cdot[(-6)\cdot(-2)+6\cdot(-3)] \gtreqless [6\cdot(-2)-6\cdot(-3)]^2$

$108 > 36$ $\Rightarrow$ extremum at P_4

$\Rightarrow -30 < 0\ ;\ -6 < 0$ $\Rightarrow$ maximum at P_4

Example 2: A manufacturer of bicycles produces two different types A and B of a bicycle. The price of a type A bicycle is $\$1,200$ per unit and the price of a type B bicycle is $\$700$ per unit. The costs of producing x units of type A and y units of type B are descripted by following cost function:

$$C(x,\, y) = 150x^2 - 100xy + 60y^2 - 400x - 500y - 10,000$$

a) Determine the production level that maximizes the bicycle manufacturer's profit.

b) What is the maximum profit in $\$$?

a) 1. set up profit function

$$P(x,\, y) = R(x,\, y) - C(x,\, y)$$

determine revenue function

$$R(x,\, y) = 1,200x + 700y$$

set up profit function

$$\begin{aligned} P(x,\, y) &= 1,200x + 700y - (150x^2 - 100xy + 60y^2 - 400x - 500y - \\ &\quad - 10,000) = \\ &= 1,200x + 700y - 150x^2 + 100xy - 60y^2 + 400x + 500y + \\ &\quad + 10,000 = \\ &= 1,600x + 1,200y - 150x^2 + 100xy - 60y^2 + 10,000 \end{aligned}$$

2. necessary condition

$$P(x,\, y) \,=\, 1{,}600x + 1{,}200y - 150x^2 + 100xy - 60y^2 + 10{,}000$$

$$P'x(x,\, y) \,=\, 1{,}600 - 300x + 100y \,=\, 0$$

$$P'y(x,\, y) \,=\, 1{,}200 + 100x - 120y \,=\, 0 \,|\cdot 3$$

$$\begin{array}{ll} \text{I} & 1{,}600 - 300x + 100y = 0 \\ \text{II} & \underline{3{,}600 + 300x - 360y = 0} \quad \text{(addition method)} \\ \text{I + II} & 5{,}200 \qquad\qquad - 260y = 0 \end{array}$$

solve for y

$$\begin{aligned} 5{,}200 - 260y &= 0 \qquad |+260y \\ 5{,}200 &= 260y \quad |:260 \\ 20 &= y \end{aligned}$$

insert result in I

$$\begin{aligned} \text{I } 1{,}600 - 300x + 100 \cdot 20 &= 0 \\ 1{,}600 - 300x + 2{,}000 &= 0 \\ 3{,}600 - 300x &= 0 \qquad |+300x \\ 3{,}600 &= 300x \quad |:300 \\ 12 &= x \end{aligned}$$

→ potential extremum at $P(12|20)$

3. sufficient condition

$$P''xx = -300$$

$$P''yy = -120$$

$$P''xy = 100$$

$$G''yx = 100$$

$\Rightarrow$ identical cross derivatives

calculate determinant

$$P''xx \cdot P''yy - P''xy \cdot P''yx$$

$$-300 \cdot (-120) - 100 \cdot 100 > 0$$

$P''xx < 0 \quad P''yy < 0$ $\rightarrow P(12|20)$ maximum

The profit is maximal at a production mix of 12 units of type A and 20 units of type B.

b) Calculation of the maximum possible profit

Insert point $(12|20)$ in $P(x,\, y)$

$$\begin{aligned} P'y(x,\, y) &= 1{,}600 \cdot 12 + 1{,}200 \cdot 20 - 150 \cdot 12^2 + 100 \cdot 12 \cdot 20 - 60 \cdot 20^2 + \\ &\quad + 10{,}000 = \\ &= 31{,}600 \end{aligned}$$

The maximum profit possible is $\$31{,}600$.

10.2.3.2 Relative Extrema with m Constraints of the Function $f = f(x_1, \ldots, x_n)$ with $m < n$

$\Rightarrow$ multiplication method according to *Lagrange*[4]

previously: $f = f(x_1, \ldots, x_n)$

now: (target) function

+ constraints

$=$ model

The system of equations (model) to be solved consists of a so-called target function and of one or multiple constraints that limit the solution set of the (target) functions.

target function: $f = f(x_1, \ldots, x_n)$

constraints:

$$\begin{aligned} g_1 &= g_1(x_1, \ldots, x_n) \\ g_2 &= g_2(x_2, \ldots, x_n) \\ &\vdots \\ g_m &= g_m(x_m, \ldots, x_n) \end{aligned}$$

precondition: $m < n$

The constraints are formatted as equations. If the constraints consist of inequalities, then the method of linear optimisation (LP model) is to be chosen for the solution (cf. Chapter 8).

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[4] Joseph-Louis de Lagrange (1736 - 1813) was an Italian mathematician and astronomer.

Solution of the equation model:

1. Formation of the so-called Lagrange function

$$\begin{aligned} L(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) &+ \lambda_1 \cdot g_1(x_1, \ldots, x_n) + \\ &+ \lambda_2 \cdot g_1(x_1, \ldots, x_n) + \\ &+ \ldots \qquad\qquad\qquad\quad + \\ &+ \lambda_m \cdot g_m(x_1, \ldots, x_n) \end{aligned}$$

$\lambda_j =$ Lagrange multiplier of the j^{th} constraint, with

$\lambda_j \in \mathbb{R}$ for all j with $j = 1, \ldots, n$

2. Set first partial derivative equal to zero

$$\begin{array}{lcl} \dfrac{\partial L}{\partial x_1} \overset{!}{=} 0 & ; & \dfrac{\partial L}{\partial \lambda_1} \overset{!}{=} 0 \\ \vdots & & \vdots \\ \dfrac{\partial L}{\partial x_n} \overset{!}{=} 0 & ; & \dfrac{\partial L}{\partial \lambda_n} \overset{!}{=} 0 \end{array}$$

$\Rightarrow$ clearly definable system of equations with $(n+m)$ unknowns and $(n+m)$ equations

3. Addition, substitution or equalisation method (cf. Chapter 4.2.4)

$\Rightarrow$ x_i-coordinates of possible extrema;

$i = 1, \ldots, n$

λ_j-values with $j = 1, \ldots, m$

4. Kind of extremum

   To decide whether maxima, minima or saddle points are present at the localized spots, the corresponding function values of f must be formed and compared to neighbouring values.

5. Interpretation of the Lagrange multiplier λ_j

   λ_j specifies the amount by which the optimum of the target function (absolute) changes if the (absolute) value of the (corresponding) constraint varies by one unit (absolute).

Example: Recipe planning of animal feed

target function: $f = f(x;y;z) = x^2 + 3y^2 + 2z^2$

1 unit input contains each

$f =$ animal feed	fat	protein
$x =$ green flour	1	/
$y =$ soy meal	3	1
$z =$ whey powder	/	2

$$\left.\begin{array}{ll} x + 3y = 30 & \text{units of fat} \\ y + 2z = 20 & \text{units of protein} \end{array}\right\} \text{constraints in form of equations}$$

The two (quality) restrictions are specified externally (as demands on the feed to be optimised).

1. $$L(x,\, y,\, z,\, \lambda_1,\, \lambda_2) = \underbrace{x^2+3y^2+2z^2}_{\text{target function}} + \underbrace{\lambda_1(30-x-3y)}_{\text{1. constraint}} + \underbrace{\lambda_2(20-y-2z)}_{\text{2. constraint}}$$

Remark:

In practice, the constraints should be formed in such a way that the absolute element is positive, so that the corresponding links to λ_j can also be positive, which a posteriori facilitates the economic or technical interpretation of the resulting values (with regard to their signs).

2. $$\begin{aligned} L'_x &= 2x-\lambda_1 && \overset{!}{=} 0 && \text{(I)} \\ L'_y &= 6y-3\lambda_1-\lambda_2 && \overset{!}{=} 0 && \text{(II)} \\ L'_z &= 4z-2\lambda_2 && \overset{!}{=} 0 && \text{(III)} \\ L'_{\lambda_1} &= 30-x-3y && \overset{!}{=} 0 && \text{(IV)} \\ L'_{\lambda_2} &= 20-y-2z && \overset{!}{=} 0 && \text{(V)} \end{aligned}$$

$\Rightarrow$ 5 equations with 5 unknowns

3. from (I) $\lambda_1 = 2x$

from (III) $\lambda_2 = 2z$

insert in (II) $6y-3\cdot(2x)-2z = 0$

$6y-6x-2z = 0$ (II)

$30-x-3y = 0$ (IV)

$20-y-2z = 0$ (V)

(IV) $x = 30-3y$

(V) $z = 10-0.5y$

insert x and z in (II): $6y-6(30-3y)-2(10-0.5y)=0$

$$25y = 200$$

$$y = 8$$

insert y in (IV) and (V): $x = 6 = 30-3\cdot 8$

$$z = 6 = 10-0.5\cdot 8$$

insert y and z in λ_1 and λ_2: $\lambda_1 = 12 = 2\cdot 6$

$$\lambda_2 = 12 = 2\cdot 6$$

$\Rightarrow f(6\,|\,8\,|\,6) = 300$

4. Interpretation

   With the recipe $x = 6$ units$_{\text{green flour}}$, $y = 8$ units$_{\text{soy meal}}$ and $z = 6$ units$_{\text{whey powder}}$, the animal feed $f = f(x,\ y,\ z)$ reaches its (local) maximum taking into consideration the two restrictions that the feed must have exactly 30 units of fat and 20 units of protein.

10.2.4 Differentials of the Function $f = f(x_1,\ ...,\ x_n)$

Requirements: $n \geq 2$ and f is continuous, i.e. differentiable, at the point under consideration $(x_{10},\ \ldots,\ x_{n0})$.

Partial Differential (1st Order)

$$df_{x_i} := \frac{\partial f}{\partial x_i} dx_i \qquad i = 1,\ldots,n$$

df_{x_i} is the partial differential of the function f with respect to the independent variable x_i.

specifically valid for $f = f(x,y)$: $df_x = \dfrac{\partial f}{\partial x}\, dx$

Interpretation

dx_i measures the (partial) change of the function value of f at a certain position $(x_{10},\ \dots,\ x_{n0})$, if the coordinate of the independent variable x_i with $i = 1,\dots,n$, changes ceteris paribus by dx_i units.

Total Differential (1st Order)

$$df = \frac{\partial f}{\partial x_i}\, dx_1 \,+\, \dots \,+\, \frac{\partial f}{\partial x_n}\, dx_n$$

df is the total (complete) differential of the function f (with respect to all independent variables x_i with $i = 1, \dots, n$).

specifically valid for $f = f(x,\ y)$: $$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy$$

Interpretation

df measures the (total absolute) change of the function value of f at a certain position $(x_{10},\ \dots,\ x_{n0})$, if the coordinates of all independent variables x_i with $i = 1, \dots, n$ change by dx_i units.

Example: The input level of the production process $f = 2y^4e^x$ of currently $x_0 = 5$ units and $y_0 = 6$ units changes by plus 2 units each. Then the (output) level of the production process f changes by (approximately) 1,282,289.695 units. The calculation is based on the total differential of f:

$$df = \frac{\partial f}{\partial x}\, dx \qquad + \qquad \frac{\partial f}{\partial y}\, dy$$

$$\frac{\partial f}{\partial x} = 2y^4e^x \qquad\qquad \frac{\partial f}{\partial y} = 8y^3e^x$$

$$dx = 2 \qquad\qquad dy = 2$$

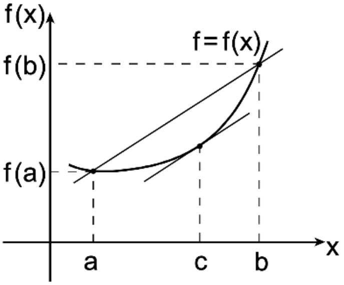
$$\Rightarrow df = 2y^4e^x \cdot 2 + 8y^3e^x \cdot 2 = 4y^4e^x + 16y^3e^x$$
$$\Rightarrow df(x_0 = 5;\ \ y_0 = 6) = 1{,}282{,}289.695 \text{ units.}$$

10.3 Theorems of Differentiable Functions

10.3.1 Mean Value Theorem for Differential Calculus

If $f = f(x)$ is continuous in the interval $[a; b]$ and differentiable in $]a; b[$, there is at least one position c with $a < c < b$, so that:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



Geometric Interpretation

Under the above mentioned conditions, there is at least one position c in $[a; b]$ where the slope of f is equal to the slope of the secant (chord) between the end points of the considered interval. The tangent of f at point c is parallel to the secant (chord) between a and b.

10.3.2 Generalized Mean Value Theorem for Differential Calculus

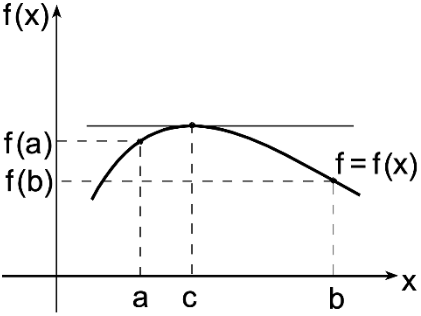
If two functions $f = f(x)$ and $g = g(x)$ are continuous in the interval $[a;b]$ and differentiable in $]a;b[$, there is at least one position c with $a < b < c$, so that:

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \qquad \text{with} \qquad g'(c) \neq 0$$

10.3.3 Rolle's Theorem[5]

If $f = f(x)$ is continuous in the interval $[a;b]$ as well as differentiable in $]a;b[$ and $f(a) = f(b)$ is also valid, there is at least one position c with $a < c < b$, so that:

$$f'(c) = 0$$



Geometric Interpretation

Under the aforementioned conditions, there is at least one position c in $[a;b]$ where the slope of f equals 0. The tangent of f at position c is parallel to the x-axis.

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[5] Michel Rolle (1652 - 1719) was a French mathematician.

10.3.4 L'Hospital's Rule[6]

If two functions $f = f(x)$ and $g = g(x)$ are continuously differentiable in $D_f = D_g$, then $x_0 \in D_f\ (= D_g)$ and $f(x_0) = g(x_0) = 0$ as well $g'(x_0) \neq 0$ are valid, so that:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

Practical Relevance

If $\lim\limits_{x \to x_0} h(x) = \lim\limits_{x \to x_0} \frac{f(x)}{g(x)}$ becomes an undefined expression, i.e. $\frac{0}{0}, \frac{+\infty}{-\infty}$ or $\frac{-\infty}{-\infty}$ (the function is divergent at these positions), the solution can be found with the help of *L'Hospital's Rule*. It is applied accordingly to the limit values $x \to +\infty$ and $x \to -\infty$.

Remark:

If $\lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)}$ again results in an undefined expression, repeat the procedure: $\lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)} = \lim\limits_{x \to x_0} \frac{f''(x)}{g''(x)}$ etc.

Example:

$$h(x) = \frac{f(x)}{g(x)} = \frac{1 - \cos x}{\sin x}$$

$$h(0) = \frac{0}{0}$$ undefined expression

$$\lim_{x \to x_0} h(x) = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$ *L'Hospital's Rule*

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[6] Guillaume François Antoine, Marquis de L'Hospital or L'Hôpital (1661 - 1704) was a French mathematician.

Remark:

Numerator and denominator are to be derived separately according to *L'Hospital's Rule*. No application of the quotient rule.

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \lim_{x \to x_0} \frac{\sin x}{\cos x} = \frac{0}{1} = 0 \qquad S_S = \{0\}$$

with $S_S =$ solution set

10.3.5 Bounds Theorem for Differential Calculus

If $f = f(x)$ is continuous in the interval $[a; b]$ as well as differentiable in $]a; b[$ and $c \leq f'(x) \leq d$ is valid, then also:

$$c(b-a) \leq f(b) - f(a) \leq d(b-a)$$

Chapter 11

Integral Calculus

11.1 Introduction

While *differential calculus* deals with the determination of the derivative (absolute gradient) $f'(x)$ of a given function $f(x)$, *integral calculus* - starting from a given derivative function $f'(x)$ - is interested in the underlying *original function* $f(x)$. The original function is called the *antiderivative*, *inverse derivative* or *primitive function*. The return from the derivative function to the antiderivative is called *integration*.

Example:

A one-product-company knows its marginal cost function:
$C'(x) = 3x^2 - 4x + 21$

The *total cost function* $C = C(x)$ is being searched for.

Thus a function $C(x)$ is searched for in such a way that its 1st derivative $C'(x)$ again represents exactly the marginal cost function $C'(x)$.

Observation of the individual summands:

(1) The 1st derivative of $f(x) = x^3$ is $f'(x) = 3x^2$.

(2) The 1st derivative of x^2 is $2x$; accordingly $-4x$ is the 1st derivative of $-2x^2$.

(3) The 1st derivative of cx is c, $c =$ constant; accordingly 21 is the 1st derivative of $21x$.

Thus one receives as (preliminary) result: $C(x) = x^3 - 2x^2 + 21x$

Check the result by using the 1st order derivation: $C'(x) = 3x^2 - 4x + 21$

F. W. Peren, *Math for Business and Economics*,
https://doi.org/10.1007/978-3-662-68860-1_11

However, there are still fixed costs, $C_f = \text{const.}$, which have to be added as well, in order to clearly determine the total cost function:

$$C(x) = x^3 - 2x^2 + 21x + C_f$$

11.2 The Indefinite Integral

11.2.1 Definition/Determining the Antiderivative

Antiderivative

f is the given continuous function in the interval $[a,b]$. A differentiable function F in $[a,b]$ is called *antiderivative* (inverse derivative, primitive function) of f, if:

$$F'(x) = f(x) \qquad \text{or} \qquad \frac{dF}{dx} = f(x).$$

Indefinite Integral

The set of all antiderivatives of f in $[a,b]$ is called the *indefinite integral*:

$\int f(x)dx = F(x) + c$ with $F'(x) = f(x)$; $c = \text{const.}$; $c \in \mathbb{R}$

Examples: What is the *indefinite integral* for:

(1) $f(x) = x^2$

$$\Rightarrow \int f(x)dx = \int x^2 dx = \frac{1}{3}x^3 + c$$

since $F'(x) = \dfrac{dF(\frac{1}{3}x^3)}{dx} = x^2$

(2) $f(x) = 4x^3$

$$\Rightarrow \int f(x)dx = \int 4x^3 dx = x^4 + c$$

since $F'(x) = 4x^3$

(3) $f(x) = ax^2 + bx + q$

$$\Rightarrow \int f(x)dx = \int (ax^2 + bx + q)dx = \frac{ax^3}{3} + \frac{bx^2}{2} + qx + c$$

since $F'(x) = \frac{3ax^2}{3} + \frac{2bx}{2} + q = ax^2 + bx + q$

(4) $f(t) = t^2 \cdot \sqrt[3]{t}$

$$\Rightarrow \int f(t)dt = \int (t^2 \cdot \sqrt[3]{t})dt = \int (t^2 \cdot t^{\frac{1}{3}})dt$$

$$= \int t^{\frac{6}{3}+\frac{1}{3}} dt = \int t^{\frac{7}{3}} dt = \frac{t^{\frac{10}{3}}}{\frac{10}{3}} + c = \frac{3}{10} \cdot \sqrt[3]{t^{10}} + c$$

since $F'(t) = \frac{\frac{10}{3} \cdot t^{\frac{7}{3}}}{\frac{10}{3}} = t^{\frac{7}{3}}$

$f(x)$	$\int f(x)dx$	Remarks
0	c	$c =$ const., $c \in \mathbb{R}$
x^n	$\frac{x^{n+1}}{n+1} + c$	$n \neq -1$ if $n \in \mathbb{N}$: $x \in \mathbb{R}$, $ax + b \in \mathbb{R}$ if $n \in \mathbb{Z}$: $x \neq 0$, $ax + b \neq \mathbb{R}$ if $n \in \mathbb{R}$: $x > 0$, $ax + b > 0$
$(ax+b)^n$	$\frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + c$	

$\frac{1}{x}$	$\ln x + c$	$x > 0$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln(ax+b)+c$	$ax+b>0, \quad a \neq 0$
e^x	$e^x + c$	$x \in \mathbb{R}$
e^{ax+b}	$\frac{1}{a}e^{ax+b}+c$	$a \neq 0$
$\sin x$	$-\cos x + c$	$x \in \mathbb{R}$
$\cos x$	$\sin x + c$	$x \in \mathbb{R}$

Examples:

(1) $\int x^7 dx = \frac{1}{8}x^8 + c$

(2) $\int dx = \int 1 dx = x + c$

(3) $\int \sqrt{y} dy = \int y^{\frac{1}{2}} dy = \frac{2}{3} y^{\frac{3}{2}} + c$

(4) $\int (2x)^4 dx = \frac{1}{2} \cdot \frac{(2x)^5}{5} + c$

(5) $\int \frac{dx}{\sqrt[5]{x^2}} = \int x^{-\frac{2}{5}} dx = \frac{5}{3} x^{\frac{3}{5}} + c$

(6) $\int (3z-2)^2 dz = \frac{1}{3} \cdot \frac{(3z-2)^3}{3} + c = \frac{1}{9}(3z-2)^3 + c$

(7) $\int\sqrt{2x-1}dx = \int(2x-1)^{\frac{1}{2}}dx = \frac{1}{2}\cdot\frac{2}{3}\cdot(2x-1)^{\frac{3}{2}}+c = \frac{1}{3}\sqrt{(2x-1)^3}+c$

(8) $\int e^{0.5t}dt = 2\cdot e^{0.5t}+c$

11.2.2 Elementary Calculation Rules for the Indefinite Integral

For integrating a function f multiplied by a constant factor and the integration of a sum of two functions $f(x) \pm g(x)$, the following rules apply:

Let f, g be continuous functions. Then the following is valid:

(1) $\int a\cdot f(x)dx = a\cdot\int f(x)dx$

(2) $\int(f(x)\pm g(x))dx = \int f(x)dx\pm\int g(x)dx$

<u>Examples:</u>

(1) $\int 6x^2dx = 6\int x^2dx = 6\cdot\frac{1}{3}x^3+c = 2x^3+c$

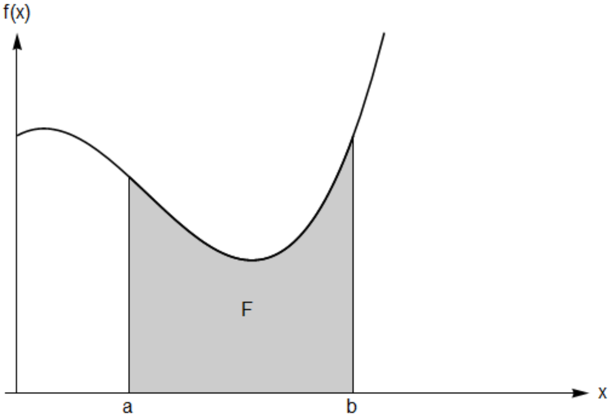
(2)
$$\begin{aligned}
&\int(8x^3-4x+2+\frac{12}{\sqrt{4x+9}})dx\\
&= \int(8x^3-4x+2)dx+\int\frac{12}{\sqrt{4x+9}}dx\\
&= \int(8x^3-4x+2)dx+12\int(4x+9)^{-\frac{1}{2}}dx\\
&= 8\frac{1}{4}x^4-4\frac{1}{2}x^2+2x+12\cdot\left(\frac{1}{4}\cdot\frac{(4x+9)^{\frac{1}{2}}}{\frac{1}{2}}\right)+c\\
&= 2x^4-2x^2+2x+12\cdot\frac{1}{4}\cdot 2\cdot(4x+9)^{\frac{1}{2}}+c\\
&= 2x^4-2x^2+2x+6\sqrt{4x+9}+c
\end{aligned}$$

11.3 The Definite Integral

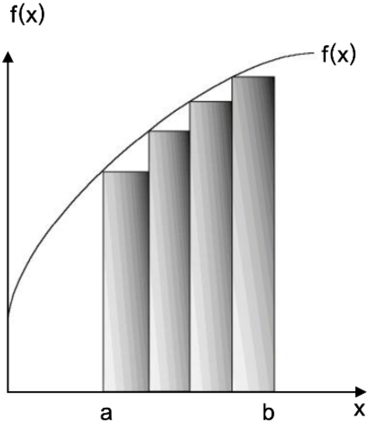
11.3.1 Introduction

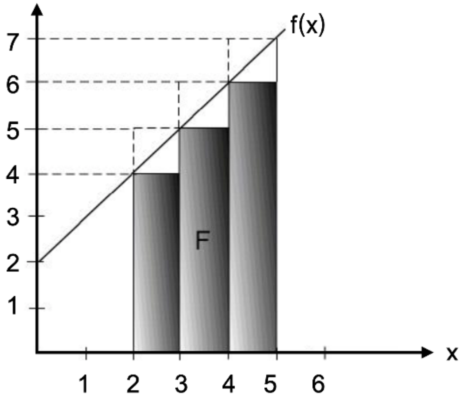
The other task of integral calculus is to determine the area F of the surface piece, which is bounded by the function graph, the x-axis and the two perpendiculars $x = a$ and $x = b$. First, the surface area, i.e. the surface measure F, of the gray marked area in the figure below shall be determined. Since not all boundary lines of the grey area are straight, elementary geometric methods fail.

Example:



The interval $[a,b]$ can be deconstructed into n arbitrary subintervals $[x_i\,;\,x_{i+1}]$ with the (variable) width $\Delta x_i = x_{i+1}\ x_i,\ \ i = 1,\ldots,n$. The area below and above the function is divided into (equal width) rectangles, whose heights are tangent to the function $f(x)$ once on the left and once on the right. The surface area below $f(x)$ in the interval $[a,b]$ then clearly lies between the sum of all surfaces of all rectangles above and the sum of all surfaces of all rectangles below $f(x)$.





To determine the area below the function $f(x)$ in the interval $[2;5]$, the interval is, for example, divided into three equal rectangles whose heights touch the graph on the left. The sum of the areas of these rectangles is: $4 \cdot 1 + 5 \cdot 1 + 6 \cdot 1 = 15$ LU^2 (LU $=$ length units).

Then the area is again divided into three rectangles of equal width, but their heights touch the graph on the right (equal abscissa intervals). Their area is: $5 \cdot 1 + 6 \cdot 1 + 7 \cdot 1 = 18$ LU^2.

The area F, which is being searched for, is between the sum of the areas of the first rectangles and the sum of the areas of the second rectangles: 15 LU$^2 < F < 18$ LU^2

If the problem, or rather the procedure, is transferred to an arbitrary, continuous function $f(x)$ over the interval $[a,b]$, the following applies:

$$\sum_{i=1}^{n} f(x_i) \cdot \Delta x_i \leq F \leq \sum_{i=1}^{n} f(x_{i+1}) \cdot \Delta x_i$$

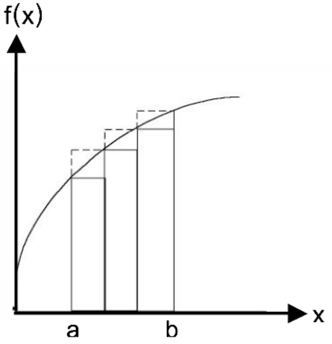
with $\Delta x_i = x_{i+1} - x_i \qquad\qquad i = 1, \ldots, n$

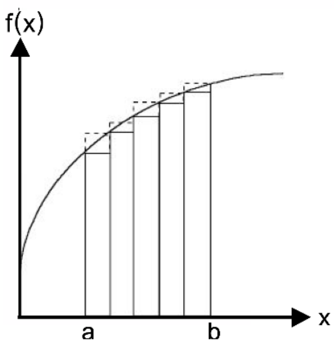
This approximation becomes more accurate the smaller the width of the intervals Δx_i is.

For the borderline case, where the width of the interval Δx_i converges towards zero ($\Delta x_i \to 0$), $f(x_{i+1})$ strives towards $f(x_i)$. The height of the rectangles below and above the graph of $f(x)$ are then nearly identical.

The sum of all areas of the rectangles above the function converges to the sum of all areas of the rectangles below the function, so that the desired area F with $\Delta x_i \to 0$ becomes more and more unambiguously - in borderline cases unambiguously - determinable.

$\Delta x_i \to 0$ with $\Delta x_i = x_{i+1} - x$	The width of the formed rectangles below and above the function $f(x)$ become smaller and smaller; the difference between the two areas converges to zero.





The smaller Δx_i is selected, i.e. the more intervals are formed, the clearer the wanted area F can be determined.

It can be determined (in borderline cases) if $\Delta x_i \to 0$ or $n \to \infty$.

$$F = \lim_{\substack{\Delta x_i \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(x_i)\Delta x_i = \lim_{\substack{\Delta x_i \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(x_{i+1})\Delta x_i = \int_a^b f(x)dx$$

The areas above and below the function $f(x)$ practically coincide. The above mentioned limiting value of a function $f(x)$, which is continuous in the interval $[a,b]$, is called *definite integral* of the function $f(x)$ in the limits a and b.

Remark:

- The definite integral $\int\limits_a^b f(x)dx$ is not a function but a fixed number. The value of the definite integral can also be negative.
- The definition of the definite integral can also be applied to discontinuous functions. So e.g. every piecewise continuous function with a finite number of jump discontinuities $x_1,\ldots,x_n$ can be integrated. The integral $\int\limits_a^b x$ is the sum of the integrals over the single function sections.

11.3.2 Relationship between the Definite and the Indefinite Integral

The value of the definite integral is equal to the difference of the values of the antiderivative of the integrand $f(x)$, $F(x)$; value of the upper limit of the antiderivative $F(x)$, $F(b)$, minus value of the lower limit, $F(a)$.

Fundamental theorem of differential and integral calculus:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

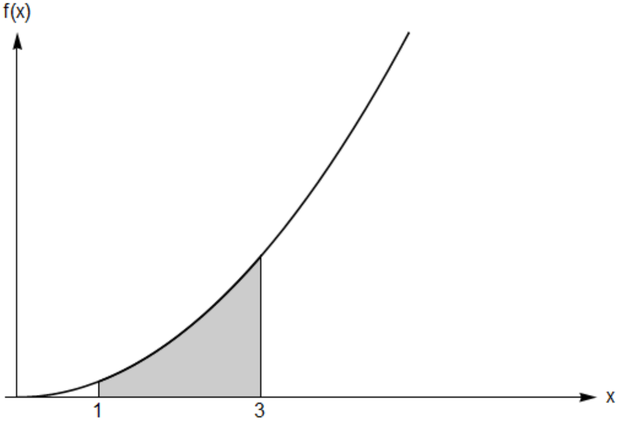
Examples:

(1) Determination of the area below the function $f(x) = x^2$ between $x_1 = 1$ and $x_2 = 3$:

$$\int_1^3 x^2 dx = \left[\frac{1}{3}x^3\right]_1^3 = \left(\frac{1}{3}3^3\right) - \left(\frac{1}{3}1^3\right) = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$$

$$= 8\frac{2}{3} \text{ AU}$$

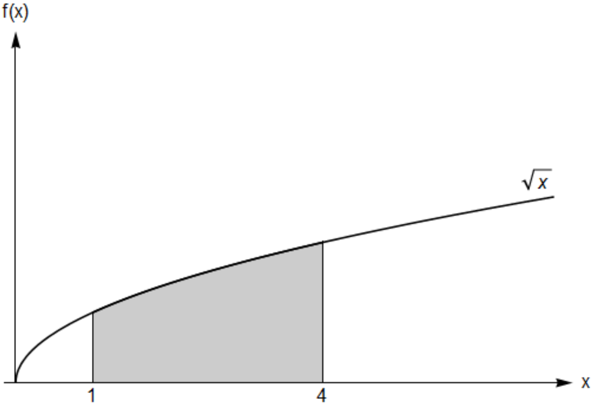
AU $=$ area units $=$ LU^2



(2) $f(x) = \sqrt{x}$ lower/upper limit: $x_l = 1,\, x_u = 4$

$$\int_1^4 \sqrt{x}dx = \int_1^4 x^{\frac{1}{2}}dx = \left[\frac{1}{\frac{3}{2}}x^{\frac{3}{2}}\right]_1^4 = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4 = \frac{2}{3}4^{\frac{3}{2}} - \frac{2}{3}1^{\frac{3}{2}} = \frac{14}{3}$$

$$= 4\frac{2}{3} \text{ AU}$$



Variation of the Upper Limit

If the lower integration limit a is kept constant and only the upper limit b is varied, there is exactly one area value F with $F = \int_a^b f(x)\,dx$ for each value of the upper limit b. That means, there is a clear relation between f and b.

To clarify this unique relationship, b is usually replaced by the independent variable x and the previous integration variable x is combined with another letter, for example t.

Thus the value F of the integral from a to the upper (variable) limit x is written as:

$$F = F(x\,;t) = \int_a^x f(t)dt \qquad t \in [a\,,x] \qquad a,x \geq 0$$

The function $F(x)$ is called *integral function* of $f(t)$ in the interval/area $[a,x]$.

Example:

$f(t) = t$

$\int\limits_a^x t dt = [F(t)]_a^x = F(x) - F(a) = \frac{1}{2}x^2 - \frac{1}{2}a^2$ with $a =$ const.

Depending on the definition of the *lower* integration limit a, the following integral functions are obtained:

$a = 0$: $\int\limits_0^x t dt = \frac{1}{2}x^2 - \frac{1}{2} \cdot 0^2 = \frac{1}{2}x^2$

$a = 2$: $\int\limits_0^x t dt = \frac{1}{2}x^2 - \frac{1}{2} \cdot 2^2 = \frac{1}{2}x^2 - 2$

$a = 10$: $\int\limits_0^x t dt = \frac{1}{2}x^2 - \frac{1}{2} \cdot 10^2 = \frac{1}{2}x^2 - 50$

Remarks:

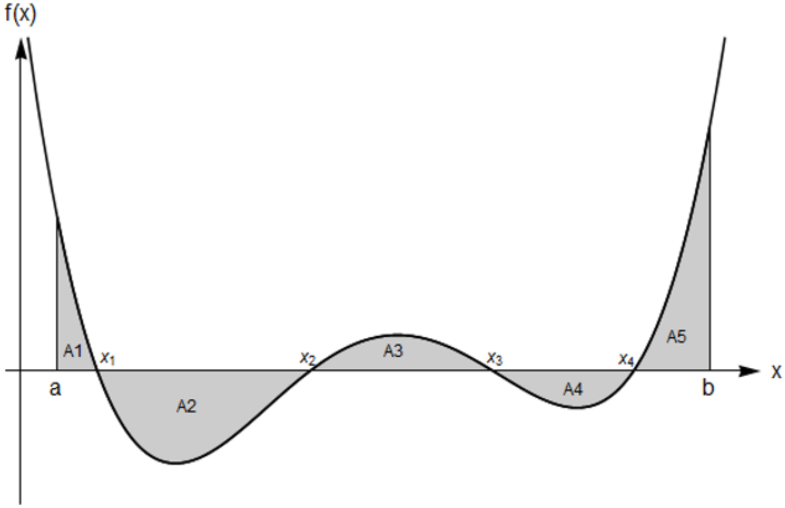
- The different integral functions of the example shown last merely differ *by an additive constant*.
- During the formation of the *definite integral*, areas, which are *above the x-axis*, are valued *positive* and those, which are *below the x-axis*, are valued *negative*, thus on balance a value of zero or less than zero can result.

Addition of the Absolute Values

$$\Rightarrow \int_a^b f(x)dx = \left|\int_a^{x_1} f(x)dx\right| + \cdots + \left|\int_{x_n}^b f(x)dx\right|$$

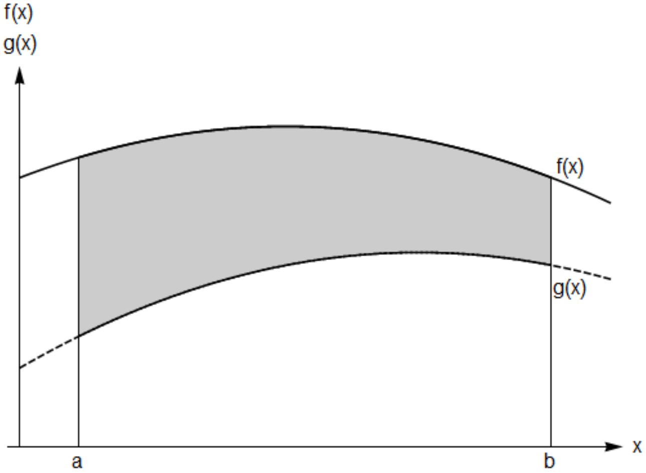
with $x_i =$ zeros of the function $f(x), \ \ i = 1, \ldots, n$

Example:



Solving the integral $\int_a^b f(x)dx$ is done by adding the absolute values of the corresponding single areas A_j, with $j = 1, \ldots, 5$.

Example:



Remark:

- The surface area of the piece of area, which is located *between two function graphs* f and g (with $f \geq g$), is calculated as the difference between the two pieces of area located below the graphs:

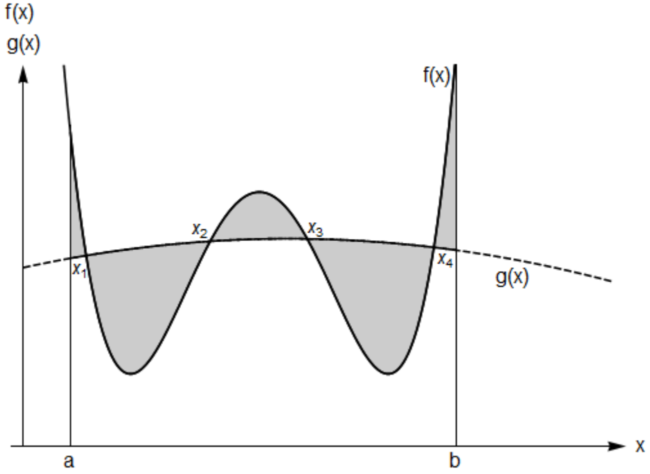
$$F(x) = \left| \int_a^b (f(x) - g(x))dx \right|$$

In case $f(x)$ and $g(x)$ intersect within $[a,b]$ with the points of intersection $x_1, x_2, \ldots, x_n$, the total area enclosed by the functions must be integrated from intersection point to intersection point to determine the total area enclosed by the functions:

$$F(x) = \left| \int_a^{x_1} (f(x) - g(x))dx \right| + \cdots + \left| \int_{x_n}^b (f(x) - g(x))dx \right|$$

with $x_i =$ point of intersection between the areas f and g, $i = 1, \ldots, n$.

To avoid negative area dimensions, the *absolute values* are used again.



11.3.3 Special Techniques of Integration

Unlike in *differential calculus*, *integration rules* do not exist for all integrable functions, i.e. there is no kind of "product rule", "quotient rule" or "chain rule".

Instead, it is attempted to transform the integrand by suitable conversions into a form, which can be integrated in closed form by using *basic integrals*.

11.3.3.1 Partial Integration

If the integrand is given as a *product*, the integral can often be transferred into a simpler form:

$$\boxed{\int f(x) \cdot g'(x) dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) dx}$$

with $f, f', g, g' =$ continuous functions

This integration technique is related to the *product rule of differential calculus*:

$$h(x) = f(x) \cdot g(x)$$
$$\Rightarrow h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

<u>Example of an **indefinite** integral:</u>

Find the solution for: $\int lnx \cdot xdx$ with $D_f = \mathbb{R}^+$

$$\Rightarrow f(x) = lnx \quad g'(x) = x \qquad \Rightarrow f'(x) = \frac{1}{x} \quad g(x) = \frac{1}{2}x^2$$
$$\Rightarrow \int lnx \cdot x\,dx = lnx \cdot \frac{1}{2}x^2 - \int \frac{1}{x} \cdot \frac{1}{2}x^2 dx$$
$$= lnx \cdot \frac{x^2}{2} - \int \frac{1}{2}xdx$$
$$= lnx \cdot \frac{x^2}{2} - \left(\frac{1}{2} \cdot \frac{1}{2}x^2 + c\right)$$
$$= lnx \cdot \frac{x^2}{2} - \frac{1}{4}x^2 - c$$

<u>Example of a **definite** integral:</u>

Find the solution for: $\int\limits_2^3 x \cdot e^x dx$

$$\Rightarrow f(x) = x \quad g'(x) = e^x \qquad \Rightarrow f'(x) = 1 \quad g(x) = e^x$$

$$\Rightarrow \int\limits_2^3 x \cdot e^x dx = [x \cdot e^x]_2^3 - \int\limits_2^3 1 \cdot e^x dx$$

$$= [x \cdot e^x - e^x]_2^3 = [(x-1)e^x]_2^3$$

$$= \left((3-1)e^3\right) - \left((2-1)e^2\right) = 2e^3 - e^2 \approx 32.78 \text{ AU}$$

11.3.3.2 Integration by Substitution

When integrating by substitution, the variable x in $\int f(x)dx$ is replaced by a suitable function $g(z)$. Provided that $g(z)$ is differentiable and reversible, the following is valid:

$$\boxed{\int f(x)dx = \int f(g(z)) \cdot g'(z)dz \text{ with } x = g(z)}$$

Example of an **indefinite** integral:

Find the solution for: $\int x\sqrt{1-x^2}dx$

$\Rightarrow$ Substitution: $1-x^2 = z \Rightarrow dz = -2x\,dx$ or $dx = -\frac{1}{2x}dz$

$$\Rightarrow \int x\sqrt{1-x^2}dx = -\frac{1x}{2x}\int\sqrt{z}\,dz = -\frac{1}{2}\int z^{\frac{1}{2}}dz$$
$$= -\frac{1}{2}\cdot\frac{1}{\frac{3}{2}}\cdot z^{\frac{3}{2}} + c$$
$$= -\frac{1}{3}\sqrt[2]{z^3} + c$$

$\Rightarrow$ Resubstitution: $\int x\sqrt{1-x^2}dx = -\frac{1}{3}\sqrt{(1-x^2)^3} + c$

Example of a **definite** integral:

Find the solution for: $\int\limits_1^2 x^3\sqrt{x^4-1}dx$

$\Rightarrow$ Substitution: $z = x^4 - 1 \Rightarrow dz = 4x^3dx \quad$ or $\quad dx = \frac{1}{4x^3}dz$

The original transformation limits $x_l = 1$ and $x_u = 2$ transform accordingly:

$$z_l = g(x_l) = 1^4 - 1 = 0$$

$$z_u = g(x_u) = 2^4 - 1 = 15$$

$$\Rightarrow \int_1^2 x^3\sqrt{x^4-1}dx = \int_0^{15} x^3\sqrt{z}\cdot\frac{1}{4x^3}dz = \int_0^{15}\frac{1}{4}\sqrt{z}dz$$

$$= \int_0^{15}\frac{1}{4}z^{\frac{1}{2}}dz = \left[\frac{1}{4}\cdot\frac{1}{\frac{3}{2}}\cdot z^{\frac{3}{2}}\right]_0^{15}$$

$$= \frac{1}{6}\cdot 15^{\frac{3}{2}} - \frac{1}{6}\cdot 0^{\frac{3}{2}} \approx 9.68 \text{ AU}$$

Resubstitution: $\left[\frac{1}{4}\cdot\frac{1}{\frac{3}{2}}\cdot z^{\frac{3}{2}}\right]_0^{15} = \left[\frac{1}{6}\left(x^4-1\right)^{\frac{3}{2}}\right]_1^2$

$$= \frac{1}{6}\left(2^4-1\right)^{\frac{3}{2}} - \frac{1}{6}\left(1^4-1\right)^{\frac{3}{2}} \approx 9.68 \text{ AU}$$

11.4 Multiple Integrals

A function with several independent variables $f = f(x_1,\ldots,x_n)$ can be integrated by partially integrating c.p. ($=$ if the remaining variables are constant) successively after all variables:

$$\int\cdots\iint f(x_1,x_2,\ldots,x_n)\,dx_1\,dx_2\ldots dx_n.$$

Remark:

The innermost integral symbol belongs to dx_1, the next following symbol to dx_2 and the outer symbol to dx_n.

The integration is done from inside to outside.

Example of an **indefinite** double integral:

$$\iint xy\,dx\,dy = \int\left[\frac{1}{2}x^2y + c(y)\right]dy = \frac{1}{4}x^2y^2 + C(y) + d(x)$$

Example of a **definite** double integral:

$$\int_2^5 \int_1^3 1\,dx\,dy = \int_2^5 \left([x]_1^3\right) dy = \int_2^5 (3-1)dy = \int_2^5 2dy$$

$= [2y]_2^5 = 10-4 = 6$ AU (here in three-dimensional space)

11.5 Integral Calculus and Economic Problems

The relationship between *total economic functions* and *marginal economic functions* is illustrated by means of the *definite integral*.

Remark:

By definition, *total economic functions* are always antiderivatives of the corresponding *marginal economic functions*.

11.5.1 Cost Functions

Let $C'(x)$ be the marginal cost function of the total cost function $C(x)$.

$$\Rightarrow \quad \int_0^x C'(q)dq = C(x)+C_f \quad \text{or} \quad \int_0^x C'(q)dq = C_v(x)$$

$$\Rightarrow \quad C(x) = \int_0^x C'(q)dq + C_f \quad \text{or} \quad C(x) = C_v(x)+C_f$$

with $C_v(x) =$ variable costs; $C_f =$ fixed costs

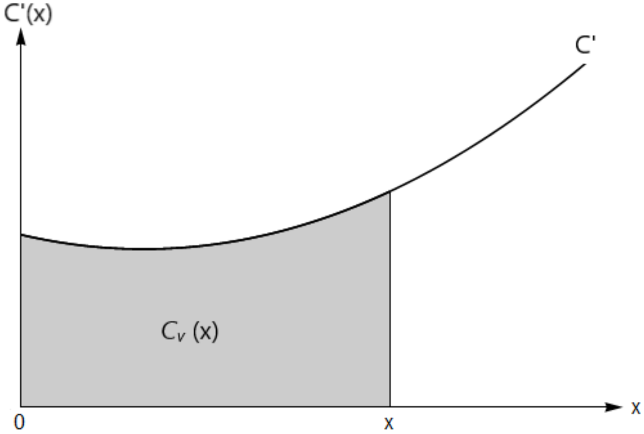
The integral of the marginal function thus corresponds to the variable costs $C_v(x)$.

The following relationships apply between the total costs $C(x)$, the marginal costs $C'(x)$, the variable costs $C_v(x)$ and the fixed costs C_f:

$$C_v(x) = \int_0^x C'(q)dq \quad \text{or}$$

$$C(x) = \int_0^x C'(q)dq + C_f$$

Graphically, the variable costs $C_v(x)$ for the output x correspond to surface area of the area below the marginal costs between zero and x.



Example:

The fixed costs C_f of $\$4,000$ and the marginal cost function $C'(x) = 0.03x^2 - 3x + 120$ [\$/unit] are known.

What are the total costs with an output x of 400 QU (quantity units)?

$$\begin{aligned}
C(x) &= C_v(x) + C_f \\
&= \int_0^{400} \left(0.03x^2 - 3x + 120\right) dx + 4,000 \\
&= \left[0.03 \cdot \frac{1}{3} \cdot x^3 - 3 \cdot \frac{1}{2} \cdot x^2 + 120x\right]_0^{400} + 4,000 \\
&= \left(0.01 \cdot (400)^3 - 1.5 \cdot 400^2 + 120 \cdot 400\right) - (0) + 4,000 \\
&= \$452,000
\end{aligned}$$

11.5.2 Revenue Function (= Sales Function)

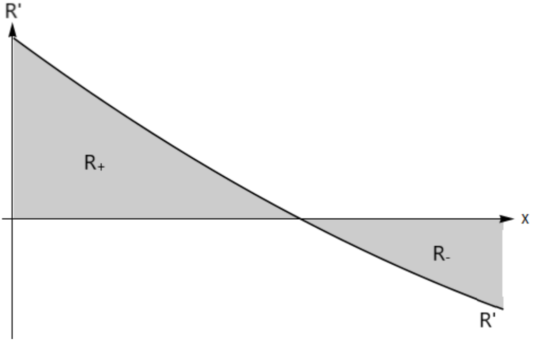
$R'(x)$ is the marginal revenue function of the revenue function $R(x)$.

$$\boxed{R(x) = \int_0^x R'(q)dq}$$

Graphically, the total revenue $R(x)$ for the quantity sold x corresponds to the surface area underneath the curve of the marginal revenue between 0 and x.

Note:

The areas located below the x-axis are **negative**.



Example:

The marginal revenue function is $R'(x) = 1{,}044 - 0.6x$ [\$/unit].

What is the revenue function and what is the associated inverse demand function (demand function)?

Revenue function:

$$R(x) = \int_0^x (1{,}044 - 0.6q)\,dq = \left[1{,}044q - 0.6 \cdot \frac{1}{2}q^2\right]_0^x = 1{,}044x - 0.3x^2 \text{ [\$]}$$

Inverse demand function:

$$R(x) = x \cdot p(x) \Leftrightarrow p(x) = \frac{R(x)}{x} = 1{,}044 - 0.3x \text{ [\$/unit]}$$

11.5.3 Profit Functions

The (total) profit $P(x)$ is determined by the difference between *revenue* $R(x)$ and total costs $C(x)$, so that:

$$\boxed{\begin{aligned} P(x) &= R(x) - C(x) = \int_0^x R'(q)dq - \left(\int_0^x C'(q)dq + C_f\right) \\ &= \int_0^x (R'(q) - C'(q))\,dq - C_f \end{aligned}}$$

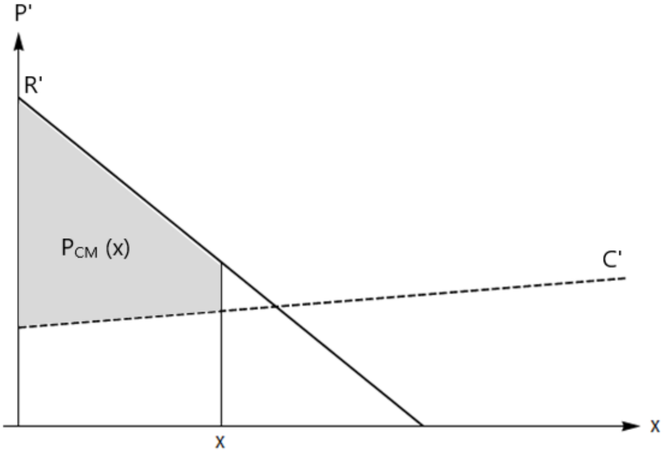
This results in the contribution margin $P_{CM}(x)$:

$$\boxed{P_{CM}(x) = \int_0^x (R'(q) - C'(q))\,dq}$$

Graphically the contribution margin $P_{CM}(x)$ is obtained for the sold quantity x as a measure of the area between the marginal revenue and the marginal cost curve.

Note:

If R' is below C', the surface pieces are evaluated **as negative**, so that the total contribution margin is the difference of the positive and negative evaluated areas.



$P_{CM}(x) =$ contribution margin for quantity x

Example:

The marginal cost function $C'(x) = 3x^2 - 24x + 60$ as well as the marginal revenue function $R'(x) = -18x + 132$ are given.
The total costs for the output of 10 QU (quantity units) amount to $\$498$.

Determine the following: (1) the revenue function,

(2) the total cost function,

(3) the inverse demand function,

(4) the profit function.

for (1): $R(x) = \int_0^x R'(q)dq = \int_0^x (-18q + 132)\,dq$

$$= \left[-18 \cdot \frac{1}{2} \cdot q^2 + 132q\right]_0^x$$

$$= -9x^2 - 132x \ \text{[\$]}$$

for (2): $C(x) = \int_0^x C'(q)dq + K_f = \int_0^x \left(3q^2 - 24q + 60\right) dq + C_f$

$$= \left[3 \cdot \frac{1}{3} \cdot q^3 - 24 \cdot \frac{1}{2} \cdot q^2 + 60q\right]_0^x + C_f$$

$$= x^3 - 12x^2 + 60x + C_f \ \text{[\$]}$$

$$\Rightarrow C(10) = \$498$$

$$\Rightarrow 10^3 - 12 \cdot 10^2 + 60 \cdot 10 + C_f = \$498$$

$$\Rightarrow C_f = 498 - 400 = \$98$$

$$\Rightarrow C(x) = x^3 - 12x^2 + 60x + 98 \ \text{[\$]}$$

for (3): $R(x) = x \cdot p(x)$

$$\Rightarrow p(x) = \frac{R(x)}{x} = \frac{-9x^2 + 132x}{x}$$

$$= -9x + 132 \ \text{[\$/unit]}$$

for (4): $P(x) = R(x) - C(x) = \left(\int\limits_0^x (R'(q) - C'(q))\, dq - C_f \right)$

$$\Rightarrow P(x) = -9x^2 + 132x - \left(x^3 - 12x^2 + 60x + 98\right)$$

$$= -x^3 + 3x^2 + 72x - 98 \ \text{[\$]}$$

Chapter 12

Elasticities

12.1 Definition of Elasticity

The subject of this chapter is the analysis of the *relative rate of change* of economic variables when there is a functional relationship between them, for example $y = y(x)$.

Absolute Changes

$\Rightarrow$ *Difference quotient* $\dfrac{\Delta y(x)}{\Delta x}$

$=$ average *absolute slope* of the function $y(x)$ in a specific interval

$\Rightarrow$ *Differential quotient* $\dfrac{df(x)}{dx}(x_0) =$ 1$^{\text{st}}$ derivative at the point x_0

$=$ slope of the function $y(x)$ at any point,
at any position x_0 that relates to an infinitesimal area around x_0

Interpretation/Question:

By how many units does the dependent variable y change when the independent variable x varies by 1 unit?

Relative Changes

By what percentage does the dependent variable change if the independent variable varies by $1\,\%$?

(1) related to a certain interval $\Rightarrow$ arc elasticity

(2) related to a certain point (at a certain location) $\Rightarrow$ point elasticity

F. W. Peren, *Math for Business and Economics*,
https://doi.org/10.1007/978-3-662-68860-1_12

12.2 Arc Elasticity

The given function is $y = y(x)$. The ratio of the relative (= percentage) changes is called arc elasticity ε_A (= the average elasticity) of y with reference to x:

$$\varepsilon_A = \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \frac{\text{relative change of } y}{\text{relative change of } x} \quad \text{with} \quad y = y(x)$$

$\Rightarrow$ the relative (= percentage) change of the dependent variable is put into relation to the relative (= percentage) change of the independent variable.

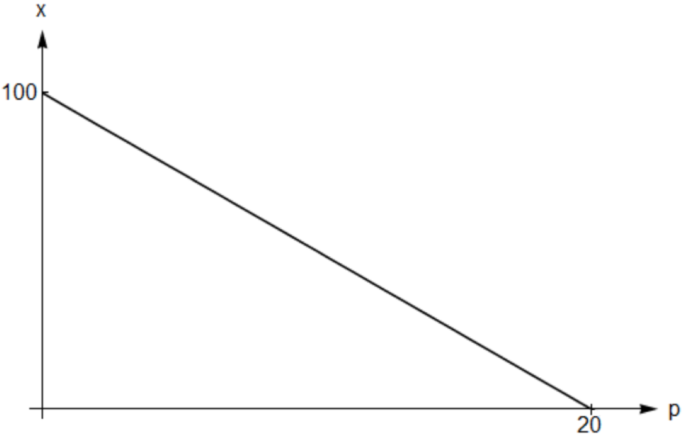
ε_A is dimensionless.

Example:

The inverse demand function is:

$p = p(x) = 20 - 0.2x$ or

$x = x(p) = 100 - 5p$ (demand function = inverse function solved for x)



General question: By what percentage does the dependent variable (here the quantity demanded of a good x) change, on *average* if the independent variable (here the price of good p) changes by one percent ?

$$\Rightarrow \left(\frac{\text{relative quantity change}}{\text{relative price change}} = \frac{\frac{\Delta x}{x}}{\frac{\Delta p}{p}} \right)$$

The quotient is called *arc elasticity* (in the considered interval/arc Δp).

Examples of absolute changes:

	Case 1	Case 2
Previous price: p	15	2
Price change: $\Delta\ p$	-1	-1
$\Rightarrow$ new price: $p+\Delta p$	14	1
Previous quantity: x	25	90
Quantity change: Δx	$+5$	$+5$
$\Rightarrow$ new quantity: $x+\Delta x$	30	95

$\Rightarrow$ In both cases, a price reduction of $\$1$ per unit determines an absolute change in demand by $+5$ QU.

$= 1^{\text{st}}$ derivative: $\quad \dfrac{dx}{dp} = -5$

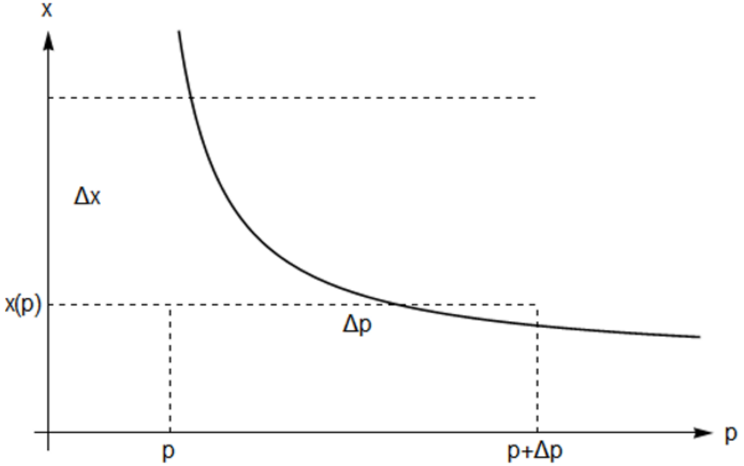
The (absolute) slope of the demand function is the same for all x. If the price is reduced by $\$1$ per unit, the quantity demanded is (constantly) reduced by 5 QU. An absolute price reduction of for example $\$15$ to $\$14$ by $\$1$ per unit is assessed relatively differently ($-6.67\,\%$) than the same absolute change about $\$1$ per unit from e.g. $\$2$ to $\$1$ per unit ($-50\,\%$).

Examples of absolute changes:

	Case 1	Case 2
price change Δp	$-6.67\,\%$	$-50\,\%$
	$= \left(\frac{14}{15} - 1\right) \cdot 100\,\%$	$= \left(\frac{1}{2} - 1\right) \cdot 100\,\%$
quantity change	$+20\,\%$	$+5.56\,\%$
	$= \left(\frac{30}{25} - 1\right) \cdot 100\,\%$	$= \left(\frac{95}{90} - 1\right) \cdot 100\,\%$
arc elasticity ε_A	$\frac{+20\,\%}{-6.67\,\%}$	$\frac{+5.56\,\%}{-50\,\%}$
	≈ -3	≈ -0.11

Example:

$$\varepsilon_A = \frac{\dfrac{\Delta x}{x(p)}}{\dfrac{\Delta p}{p}} = \frac{\dfrac{x(p+\Delta p)-x(p)}{x(p)}}{\dfrac{(p+\Delta p)-p}{p}} = \frac{\dfrac{30-25}{25}}{\dfrac{14-15}{15}} = -3$$



Interpretation of $\varepsilon_A = 3$:

If the price of the good p rises or falls by $1\,\%$, demand x falls or rises by an average of $3\,\%$ in the price range between $\$14$ and $\$15$ per unit.

Example:

$y(x) = x^2 + 1$

ε_A between $x_1 = 3$ and $x_2 = 4$?

$$x_1 = 3 \quad \wedge \quad x_2 = 4 \qquad \Rightarrow \Delta x = +1$$

$$\Rightarrow \qquad y(x_1) = y(3) = 3^2 + 1 = 10$$

$$y(x_2) = y(4) = 4^2 + 1 = 17$$

$$\Rightarrow \qquad \Delta y = +7$$

$$\Rightarrow \varepsilon_A = \frac{\frac{\Delta y}{x}}{\frac{\Delta x}{x}} = \frac{\frac{+7}{10}(= +70\%)}{\frac{+1}{3}(= +33\%)} \approx 2.1$$

i.e.: If the independent variable x increases or decreases by $1\,\%$, the value of the function y rises or falls on average by about $2.1\,\%$ in the interval between $x_1 = 3$ to $x_2 = 4$.

12.3 Point Elasticity

While the arc elasticity indicates the average rate of change within an interval, in economic sciences it is usually the elasticity around a certain point that is of interest, i.e. at a certain place x_0.

$\Rightarrow$ determination of the limit value of the arc elasticity for $\Delta x \to 0$

$$\Rightarrow \quad \lim_{\Delta x \to 0} \varepsilon_A = \lim_{\Delta x \to 0} \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \lim_{\Delta x \to 0} \frac{\Delta y}{y} \cdot \frac{x}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot \frac{x}{y} = \frac{dy}{dx} \cdot \frac{x}{y} = y'(x) \cdot \frac{x}{y} = \varepsilon$$

$\varepsilon(x) =$ (point) elasticity function

$\varepsilon(x = x_0) =$ (point) elasticity or, in other words, elasticity of the function $y = y(x)$ at the point $x = x_0$.

ε is dimensionless.

Definition

y is a differentiable function with the independent variable x.

This means then

$$\varepsilon(x = x_0) = \frac{\frac{dy}{y}}{\frac{dx}{x}} = \frac{dy}{dx} \cdot \frac{x}{y} = y'(x) \cdot \frac{x}{y} \qquad \text{with} \qquad x, y \neq 0$$

is called (point) elasticity ε of the function $y = y(x)$ at the point $x = x_0$. The numerical value of the (point) elasticity ε of y related to x at a specific point $x = x_0$ indicates (approximately) by what percentage the dependent variable y changes if the independent variable x (at this point x_0) varies marginally by $1\,\%$; with $y = y(x)$.

Note:

The sign of elasticity ε plays an important role (Tab. 12.1):

(1) If $\varepsilon > 0$ with $y = y(x)$ then $\dfrac{\frac{dy}{y}}{\frac{dx}{x}} > 0$ applies by definition,

i.e. the relative changes of the considered variables are either both positive or both negative. Thus a relative increase (decrease) of x causes a relative increase (decrease) of y. y and x are positively correlated.

(2) If $\varepsilon < 0$, a relative increase (decrease) of x causes a relative decrease (increase) of y. y and x are negatively correlated.

(3) If $\varepsilon = 0$, y remains constant with an increase (decrease) of x.
y and x are not correlated.

Example:

$f(x) = x^2 - x + 10$

What is the (point) elasticity of f at $x_0 = 10$?

$$\varepsilon = \varepsilon(x) = \frac{\frac{df(x)}{f(x)}}{\frac{dx}{x}} = \frac{df(x)}{dx} \cdot \frac{x}{f(x)} = f'(x) \cdot \frac{x}{f(x)} =$$

$$= \frac{(2x-1) \cdot x}{x^2 - x + 10} = \frac{2x^2 - x}{x^2 - x + 10} = \text{elasticity function}$$

$$\varepsilon(10) = \frac{2 \cdot 10^2 - 10}{10^2 - 10 + 10} = 1.9 = \text{(point) elasticity at } x = 10$$

Interpretation:

If x is increased (decreased) by $1\,\%$ at the position $x = 10$, the value of the function $f(10)$ will then be increased (decreased) by $1.9\,\%$ (disproportionately). The relation between x and $f(x)$ is “elastic”. $f(x)$ and x are positively correlated.

Value of Elasticity	**General Definition**	**Example:** **Demand Function** $x = x(p)$ with $p =$ price and $x =$ quantity
$\mid \varepsilon_{xp} \mid < 1$	x is inelastic	Relatively low reaction of the consumer to price changes.
$0 < \varepsilon < 1$ or $-1 < \varepsilon < 0$	(x changes relatively less than p)	Example: goods that can hardly be substituted: bread, medicine.
$\mid \varepsilon_{xp} \mid > 1$	x is elastic	Relatively strong reaction of the consumers to (small) relative price changes.
$\varepsilon > 1$ or $\varepsilon > -1$	(x changes relatively stronger than p)	Example: substitutable goods
Special case: $\mid \varepsilon_{xp} \mid = 1$	x is proportionally elastic; isoelastic	A price change of $1\,\%$ causes a proportional quantity change of $1\,\%$.
$\varepsilon = 1$ or $\varepsilon = -1$	(the relative changes of x and p are equal)	
Borderline case: $\mid \varepsilon_{xp} \mid \to \infty$	x is perfectly elastic	Borderline case: The price is constant, regardless of the level of demand.
$\varepsilon \to \infty$ or $\varepsilon \to -\infty$	(x reacts infinitely strong to small relative changes of p)	Example: fixed-price substitutable goods (branded articles in the polypoly).

Borderline case: $\varepsilon = 0$	x is perfectly inelastic; rigid (x does not react to insignificant relative changes of p)	Borderline case: The demand is constant, i.e. independent of the price. <u>Example:</u> indispensable goods such as essential medicines.

Tab. 12.1: Elasticities | Case Distinction

12.4 Price Elasticity of Demand ε_{xp}

Definition

The *price elasticity of demand* ε_{xp} measures the relative change in demand in consequence of a relative change of the price by $1\,\%$ at a specific point $(x_0|p_0)$.

$$\varepsilon_{xp} = \frac{\frac{dx}{x}}{\frac{dp}{p}} = \frac{dx(p)}{dp} \cdot \frac{p}{x}$$

Attention:

- dependent varibale = quantity demanded x
- independent variable = price p

$\Rightarrow$ inverse demand function: $x = x(p)$

<u>Note:</u>

If the inverse demand function $p = p(x)$ is used, the *price elasticity of demand* ε_{xp} would be required analogously.

Example:

Given, the inverse demand function is $p(x) = 10 - 0.5x$. Searched is the price elasticity of demand at a price of $p_0 = \$6$/QU.

Since in this case, p is the independent variable and x the dependent variable, the demand function $x = x(p)$ must be formed.

$$\Rightarrow \quad x(p) = \frac{10 - p}{0.5} = 20 - 2p$$

$$\varepsilon_{xp} = \frac{\dfrac{dx}{x}}{\dfrac{dp}{p}} = \frac{dx(p)}{dp} \cdot \frac{p}{x} = x'p(x) \cdot \frac{p}{x(p)}$$

$$x(p) = 20 - 2p$$

$$x'(p) = -2$$

$$p_0 = \$6\text{/QU}$$

$$x(6) = 20 - 2 \cdot 6 = 8 \text{ QU}$$

$$\varepsilon_{xp} = -2 \cdot \frac{6}{8} = -1.5$$

Interpretation:

With regard to the basic price $p_0 = \$6$/QU, an increase (decrease) in price by $1\,\%$ causes a demand decrease (increase) by approximately $1.5\,\%$. Consumers react elastically at a price of $p_0 = \$6$/QU ($|\ \varepsilon_{xp}\ | > 1$).

Example:

Given, the inverse demand function is $p(x) = 10 - 0.5x$. The price or quantity intervals are to be found, where the demand is

(a) elastic,

(b) inelastic,

(c) proportionally elastic (isoelastic),

(d) perfectly inelastic,

(e) perfectly elastic.

$$\varepsilon_{xp} = \frac{\frac{dx}{x}}{\frac{dp}{p}} = \frac{dx(p)}{dp} \cdot \frac{p}{x}$$

$$\begin{aligned}
\Rightarrow \quad & x(p) = ? \\
& p = 10 - 0.5x \\
\Leftrightarrow \quad & 0.5x = 10 - p \quad \Leftrightarrow \quad x(p) = 20 - 2p
\end{aligned}$$

(a) The demand is **price elastic** if the following applies: $\varepsilon_{xp} < -1$. Since the slope of the inverse demand function is negative (quantity and price are here negatively correlated), the case of elasticity is not applicable. Hence, $\varepsilon_{xp} > 1$.

$$\varepsilon_{xp} = x'(p) \cdot \frac{p}{x(p)} = -2 \cdot \frac{p}{20 - 2p} = \frac{-p}{10 - p} < -1$$

$$\begin{aligned}
\Leftrightarrow & -p \ < -1(10 - p) \\
\Leftrightarrow & -p \ < -10 + p \\
\Leftrightarrow & -2p < -10 \\
\Leftrightarrow & \ \ 2p > 10 \\
\Leftrightarrow & \ \ \ p \ > \$5/\text{QU}
\end{aligned}$$

Interpretation:

The demand is *price elastic* for prices between $\$5$ and $\$10$ per unit. The corresponding quantity range is between 0 and 10 units.

(b) The demand is **price inelastic**, if: $\varepsilon_{xp} > -1$

$$\begin{aligned}
\varepsilon_{xp} &= \frac{-p}{10-p} > -1 \\
\Leftrightarrow -p &> -1(10-p) \\
\Leftrightarrow -p &> -10+p \\
\Leftrightarrow -2p &> -10 \\
\Leftrightarrow \quad 2p &< 10 \\
\Leftrightarrow \quad p &< \$5/\text{QU}
\end{aligned}$$

Interpretation:

The demand is *price inelastic* for prices between $\$0$ and $\$5$ per unit. The corresponding quantity interval is between 10 and 20 units.

(c) The demand is **isoelastic**, if: $\varepsilon_{xp} = -1$

$$\begin{aligned}
\varepsilon_{xp} &= \frac{-p}{10-p} = -1 \\
\Leftrightarrow p &= \$5/\text{QU} \\
\Rightarrow x &= 10 \text{ QU}
\end{aligned}$$

Interpretation:

The demand is *isoelastic* if the price is $\$5$ per unit. If that price changes by $1\,\%$ ($\$4.95$ or $\$5.05$), a proportional quantity change of $1\,\%$ (9.90 QU or 10.10 QU) is affected.

(d) The demand is **perfectly price inelastic**, if: $\varepsilon_{xp} = 0$

$$\varepsilon_{xp} = \frac{-p}{10-p} = 0$$

$\Leftrightarrow p \rightarrow \$0$/QU

$\Rightarrow x = 20$ QU

Interpretation:

If the price converges towards zero or even becomes zero, all 20 QU are sold. Consumers do not react at all to (marginal) relative price changes.

(e) The demand is **perfectly price elastic**, if: $\varepsilon_{xp} = \infty$ or $\varepsilon_{xp} = -\infty$

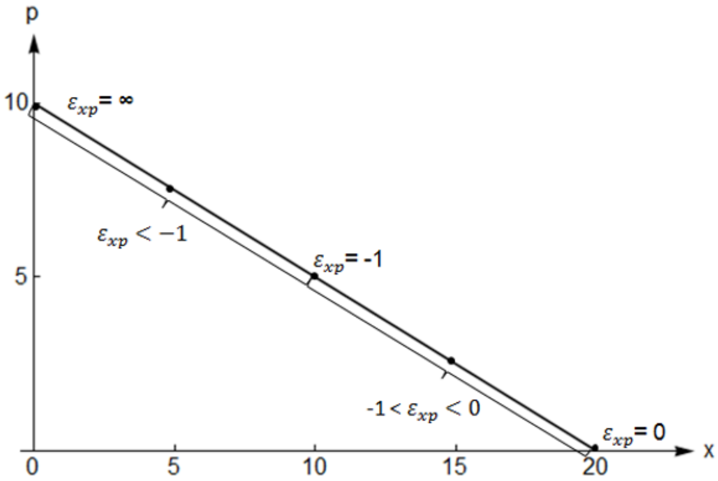
$$\varepsilon_{xp} = \frac{-p}{10-p} \rightarrow -\infty$$

$\Leftrightarrow p \rightarrow \$10$/QU

$\Rightarrow x = 0$ QU

Interpretation:

If the price converges towards $\$10$/QU or even becomes $\$10$/QU, nothing can be sold. Consumers react perfectly elastic to (marginal) relative price changes.



12.5 Cross Elasticity of Demand $\varepsilon_{x_A p_B}$

The quantity demanded x_A of a good A depends not only on the (relative) changes of its own price p_A but also on the price changes of other goods, such as good B: $x_A = x_A(p_A, p_B)$.

This relationship is known as the *cross elasticity* of demand (or cross-price elasticity of demand), $\varepsilon_{x_A p_B}$:

$$\varepsilon_{x_A p_B} = \frac{\text{relative change in quantity of good A}}{\text{relative change in price of good B}}$$

Related to an infinitesimal range at a specific point $x_A = x_A(p_A, p_B)$, the following applies:

$$\varepsilon_{x_A p_B} = \frac{\frac{\partial x_A}{x_A}}{\frac{\partial p_B}{p_B}} = \frac{\partial x_A}{\partial p_B} \cdot \frac{p_B}{x_A(p_A, p_B)}$$

Example:

The demand x_α for the type of notebook “Alpha” depends on the price p_α of its own system and the price p_β of the competitors’ system “Beta”. The corresponding demand function is:

$$x_\alpha = 10,000 - 2p_\alpha + 3p_\beta$$

The system prices are currently at:

$p_\alpha = \$2,000$ per unit α and $p_\beta = \$2,200$ per unit β

What is the cross elasticity of α in relation to the price of β at the current price situation?

$$\begin{aligned}\varepsilon_{x_\alpha p_\beta} &= \frac{\dfrac{\partial x_\alpha}{x_\alpha}}{\dfrac{\partial p_\beta}{p_\beta}} = \frac{\partial x_\alpha}{\partial p_\beta} \cdot \frac{p_\beta}{x_\alpha(p_\alpha, p_\beta)} = \\ &= 3 \cdot \frac{p_\beta}{10,000 - 2p_\alpha + 3p_\beta} = \frac{3 \cdot 2,200}{10,000 - 2 \cdot 2,000 + 3 \cdot 2,200} \approx 0.5238\end{aligned}$$

Interpretation:

A price increase (decrease) of the system “Beta” by $1\,\%$ causes a demand increase (decrease) for “Alpha” by $0.52\,\%$. x_α and p_β are positively correlated, however in an inelastic case. There is no significant substitution effect.

12.6 Income Elasticity of Demand ε_{xy}

The quantity demanded x of a good depends on the changes of the consumers' income y: $x = x(y)$. This relationship (regarding relative changes of the relevant variables) is known as the income elasticity of demand ε_{xy}.

$$\varepsilon_{xy} = \frac{\frac{dx}{x}}{\frac{dy}{y}} = \frac{dx}{dy} \cdot \frac{y}{x(y)} = x'(y) \cdot \frac{y}{x(y)}$$

Example:

The saleable quantity x of a certain vehicle depends on the net monthly income y in relation to a certain region. The consumption function is: $x = 5,500 + 50y$

Last year, the total sales in that region were $100,500$ vehicles and the average net monthly income was $\$2,600$.

What was the income elasticity of demand?

$$\varepsilon_{xy} = x'(y) \cdot \frac{y}{x(y)} = \frac{50 \cdot 2,600}{5,500 + 50 \cdot 2,600} \approx +0.96$$

Interpretation:

An increase of the net income by $1\,\%$ ($= \$26$) results in an increase in demand of that vehicle by $0.96\,\%$. This indicates an isoelastic case.

Chapter 13

Economic Functions

13.1 Supply Function

The supply function represents the relationship between the **market price of a good** (independent variable) and the **quantity supplied** (dependent variable) in the form of a unique graph (function).

Typically, a high price indicates a large quantity of goods supplied. If the price falls, the quantity offered is also usually reduced. This is depicted by the *supply function*

$$x = x(p) \text{ with } x = \text{quantity supplied and } p = \text{offer price.}$$

An analogous explanation can be given for the *inverse function* $p = p(x)$, the **inverse supply function** (Fig. 13.1). A large supply exists at a high, realizable price. The supply is small when the prices of this tradable good or service offered are low and thus unattractive for the supplier.

In macroeconomics, a distinction is made between an **individual supply function** and the **aggregate of individual supplies**. If the state of *perfect competition* exists, the marginal costs correspond to the offer price and the marginal cost function coincides with the supply function.

The supply function runs **strictly monotonically increasing**. It can be linear, as shown in the figure, or partially or completely curved (concave or convex). If the supply function is flat, the *price elasticity of the supply* or *price elasticity of the suppliers* is expected to be high. Relative price changes then cause disproportionately high, relative quantity changes. This means that the reaction of suppliers to price changes is disproportionately strong. A (tendentially) steep course of the supply function indicates a relatively low *supply elasticity*.

F. W. Peren, *Math for Business and Economics*,
https://doi.org/10.1007/978-3-662-68860-1_13

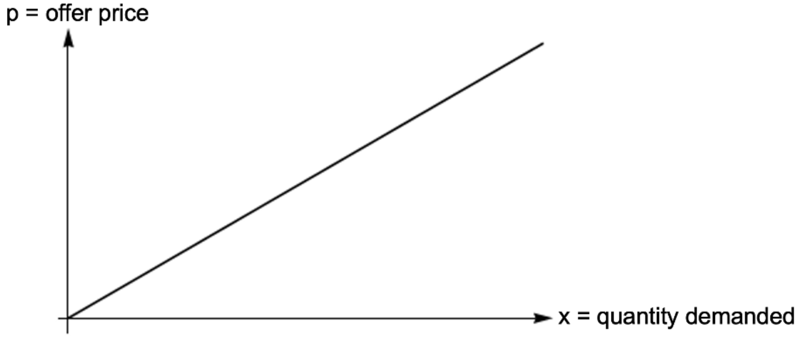


Fig. 13.1: Inverse Supply Function

Example 1:

The (inverse) supply function, which reflects the behaviour of suppliers in relation to price, is: $x(p) = 4p - 10$.

If one wants to estimate the quantity supplied at a currently prevailing price of $\$20$, the following calculation is to be made:

$x(20) = 4 \cdot 20 - 10 = 70$ QU QU = quantity unit(s)

With an offer price of $\$20$, the corresponding quantity supplied that is offered is 70 QU.

Example 2:

Suppliers' behaviour of a manufacturing firm can be described using the supply function $p(x) = 12.5x + 4$.

The corresponding offer price for a quantity supplied of 5 QU shall be calculated as follows:

$p(5) = 12.5 \cdot 5 + 4 = \$66.5$

With a quantity supplied of 5 QU, the offer price is $\$66.5$.

13.2 Demand Function / Inverse Demand Function

The *demand function* $x(p)$ represents the quantity demanded x (QU) for a good or a service depending on the market price p (\$/QU $=$ \$ per unit):

$x = x(p)$ with $x =$ quantity demanded and $p =$ asking price

The inverse demand function $p(x)$ is the inverse function of a demand function: $p(x) = f^{-1}(x(p))$. $p(x)$ views the price p as a function of quantity x. The inverse demand function $p(x)$ treats the price as a function of quantity demanded. It is also called the price function.

In contrast to the supply function, the inverse demand function (Fig. 13.2) is usually strictly monotonically decreasing. If the market price rises/falls, the quantity demanded falls/increases.

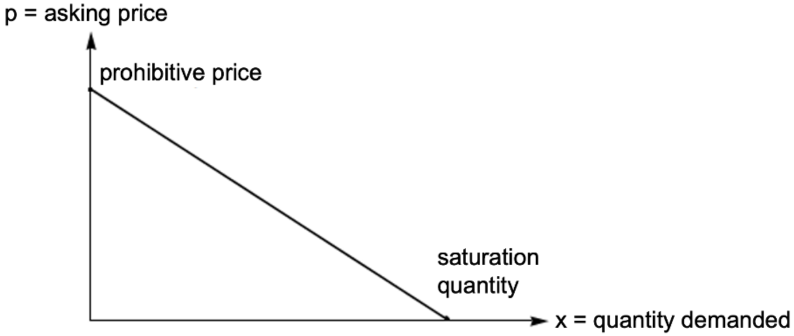


Fig. 13.2: Inverse Demand Function

For certain goods or services, e.g. luxury goods or services, the relationship between demanded quantity and price can also be reversed, i.e. when the price rises/falls, quantity demanded rises/falls (*Giffen's paradox, snob effect*).

Similar to the *supply function*, a distinction is made between an individual and an aggregated demand function.

The graphical representation of the inverse demand function is marked by two significant points: the *saturation quantity* and the *prohibitive price*.

The *saturation quantity* is determined at a price of $\$0$ per unit; in the graph, the saturation quantity corresponds to the intersection of the inverse demand function with the x-axis. This corresponds to the highest possible quantity demanded for a good or service.

Analogously, the *prohibitive price* is determined by setting the quantity demanded to zero. This state occurs when the price is so high that no one demands this good or service. Graphically, the amount of the prohibitive price is determined by the y-intersect of the inverse demand function.

Example 1:

The demand function for a manufacturer of coffee cups is $x(p) = -2p+7$. If the price per cup is set at $\$3$, the quantity demanded is determined as follows:

$x(3) = -2 \cdot 3 + 7 = 1$ QU

One coffee cup is demanded at $\$3$.

Example 2:

If the prohibitive price of the inverse demand function is $p(x) = -0.2x + 10$, the quantity demanded x is equated to zero and the following is obtained:

$p(0) = -0.2 \cdot 0 + 10 = \$10$/QU

At a price of $\$10$, no quantity is demanded.

Example 3:

For the inverse demand function $p(x) = -0.25x + 17$, the direction of causality is changed. $p(x) = -0.25x + 17$ must be calculated in terms of x as follows:

$$p - 17 = -0.25x$$

$$\frac{p - 17}{-0.25} = x$$

$$-4p + 68 = x$$

$$x = x(p) = -4p + 68$$

13.3 Market Equilibrium

When the supply and demand of a good or service in the market coincide, the market is in equilibrium. In the *market equilibrium* (Fig. 13.3) the quantity supplied is completely demanded or sold on the market. The market is “cleared”. Therefore, it is also called *market clearing*.

The equilibrium price and the equilibrium quantity represent the perfect market. The quantity supplied corresponds exactly to the demanded quantities of the consumers. There is no waste of this good. Also, a further exchange would not be an improved alternative. In reality, this state is often not reached due to insufficient transparency. If goods or services are (also) traded online, transparency is usually improved, which in turn should bring trade closer to market equilibrium.

Graphically, the market equilibrium is described by the intersection of the supply and inverse demand function.

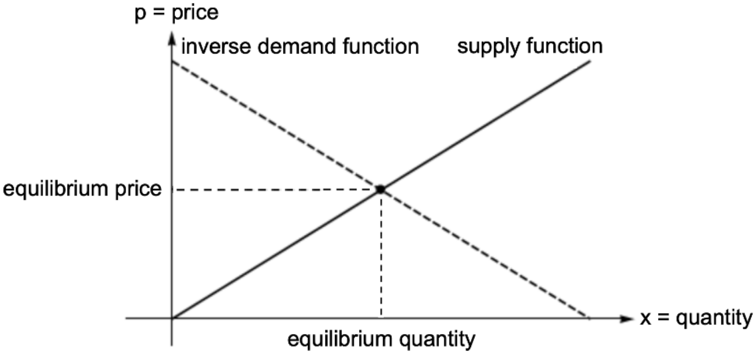


Fig. 13.3: Market Equilibrium

13.4 Buyer's Market and Seller's Market

The position in which a buyer or seller is situated determines the market situation in comparison to the market equilibrium.

If supply is greater than demand at a given price, the buyer is in a relatively better position than the seller. The *market surplus* defines the market as the *buyer's market*. A market surplus exists when the supply of a good or service exceeds the demand. The supplier(s) can manage this imbalance to their benefit by reducing their supply and minimising their surplus.

Likewise, a *market shortage* leads to a *seller's market*. If the demand is greater than the supply, this strengthens the position of the seller in relation to the buyer. In a monopolistic market, higher prices can usually be set so that the seller can dominate the market. Also, a densification of the market as a result of mergers can lead to the fact that substitute goods are rarely or never found on the market. Even in the case of emergency goods, such as medicines, a certain (temporary) dependence of the buyer on the seller can arise and shift the price away from the market equilibrium at the expense of the buyer. Other reasons for the occurrence of a seller's market can be an overregulated market, paucity of information of the buyer or a lack of competition.

13.5 Supply Gap

A *supply gap* exists if the quantity supplied is smaller than the quantity demanded for a particular good or service in a specific market.

13.6 Demand Gap

Unlike the supply gap, the *demand gap* is a phenomenon where the quantity supplied is greater than the quantity demanded for a good or service in a specific market.

Both the supply gap and the demand gap refer to an *imperfect market condition*.

Example 1:

The market equilibrium between supply and demand should be identified. The supply can be described by the supply function $p(x) = 0.5x+9$ and the demand by the inverse demand function $p(x) = -0.75x+13$. Thus, the point is being searched, i.e. the quantity and the price, where the supply function and the inverse demand function intersect:

supply function $=$ inverse demand function

$0.5x+9 = -0.75x+13$

$-4 = -1.25x$

$x = 3.2$ QU

$p(3.2) = 0.5 \cdot 3.2 + 9 = \$10.6$/QU

The market equilibrium is located at the point $(3.2|10.6)$.

Example 2:

Supposedly, the inverse demand function shifts due to an increase in income. The question to be discussed is how the initial equilibrium consequently develops.

The inverse demand function is shifted upwards to the right in response to the increased income because at any given price p, a larger quantity x can now be demanded. A new point $\big(p(x_0)|x_0\big)$ arises which describes the new market equilibrium.

Example 3:

To protect a firm's production, a lower price limit of $\$5$/QU is implemented. With the inverse demand function $p_D(x) = -6x + 17$ and the supply function $p_S(x) = 2x - 3$, it is to be determined whether a price of $\$5$/QU leads to a surplus/a demand gap or to a shortage/a supply gap.

The fixed price $\$5$/QU can be placed in both functions:

Inverse demand function: $5 = -6x + 17 \quad \rightarrow x_D = 2$ QU

Supply function: $5 = 2x - 3 \quad \rightarrow x_S = 4$ QU

With a lower price limit of $\$5$/QU, the supply is greater than the demand. In this case, there is a surplus/a demand gap of 2 QU.

13.7 Revenue Function

The *revenue function* describes the behaviour of the revenue R in relation to the quantity sold x. The quantity sold x depends on the price p, which can either be constant or vary depending on the quantity $p = p(x)$. If the inverse demand function $p(x)$ is multiplied by the quantity x, the revenue function is obtained depending on the quantity x:

$R(x) = p(x) \cdot x$ with $p =$ selling price
$x =$ quantity purchased / sold

Similarly, the revenue function can also be modeled in relation to the price p:

$R(p) = x(p) \cdot p$ with $p =$ selling price
$x =$ quantity purchased / sold

Two different cases must be distinguished:

1. The price p is constant

The quantity sold x does not change in response to price alterations (Fig. 13.4). It is not influenced by discounts or other price alterations. There is no causal relationship between the selling price p and quantity sold x. The revenue ($R = R(x) = p \cdot x$) is proportional to the quantity. In other words, the revenue function is linear. The slope of the revenue function, i.e. the first derivative of $R(x)$, corresponds to the constant price p. The intercept is zero. If no sales are generated ($x = 0$) the revenue is also zero. A relative maximum in revenue does not exist because the revenue grows constantly, i.e. linearly, along with the quantity sold x, so that the maximum is at the right edge of the revenue function, i.e. at $x = x_{max}$. The minimum in revenue is zero and is located at the origin of the revenue function at the point $(0|0)$ where there are no sales generated, i.e. at $x = x_{min} = 0$. If the good or service is not sold, no revenue is generated, $R_{min} = R(0) = 0$.

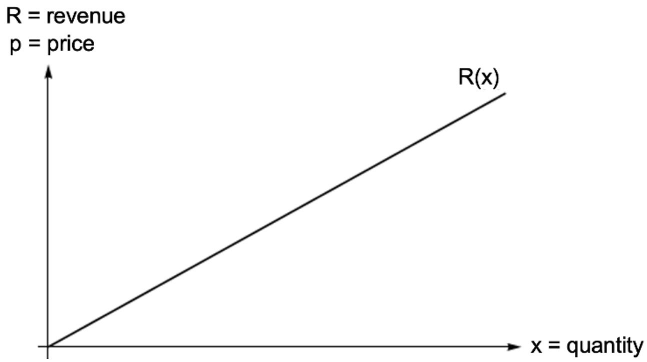


Fig. 13.4: Behaviour of Revenues Based on a Linear Revenue Function

2. The price $p = p(x)$ is variable

The quantity sold $x = x(p)$ changes due to price alterations p (Fig. 13.5). There is a causal relationship between the selling price $p = p(x)$ and the quantity sold $x = x(p)$. The revenue $R = R(x) = p(x) \cdot x$ is now generated according to the inverse demand function at variable prices with $p(x) = -mx + b$. This is modeled by a quadratic revenue function:

$$R(x) = p(x) \cdot x = (-mx + b) \cdot x = -mx^2 + bx$$

The revenue function resembles a parabola opening downwards that starts at the origin $(0|0)$ and ends at the point $(x_{max}|0)$. x_{max} represents the *saturation quantity*, i.e. the maximal quantity sold at a price of $p_{min} = 0$.

The two zeros are calculated as follows:

$$-mx^2 + bx = 0$$

$$x(-mx + b) = 0$$

$$x = 0 \lor -mx + b = 0$$

$$x = 0 \lor x = \frac{b}{m}$$

The relative maximum of the revenue function is calculated as follows:

$$R(x) = -mx^2 + bx$$

$$dR(x)/dx = -2mx + b = 0$$

$$x = \frac{b}{2m}$$

$$d^2R(x)/dx^2 = -2m < 0$$

$$\rightarrow \text{the maximum is at } x = \frac{b}{2m}$$

The maximum in revenue (maximum of the revenue function) is located at the point with the coordinates

$$x = \frac{b}{2m} \text{ and } R\left(\frac{b}{2m}\right) = -m\left(\frac{b}{2m}\right)^2 + b\left(\frac{b}{2m}\right) = \frac{-b^2}{4m} + \frac{b^2}{2m} .$$

The first zero is derived from a quantity sold of $x_{min} = 0$. The second zero at $x_{max} = \frac{b}{m}$ describes the scenario where the price $p = p_{min} = 0$, i.e. the *saturation quantity* is reached. The market is flooded with this good or service to such an extent that the price becomes zero. The quantity inflates until it reaches the price $p = 0$. The *saturation quantity* is the quantity that is reached at a price of 0. Half of the saturation quantity is the quantity sold that generates the maximum revenue.

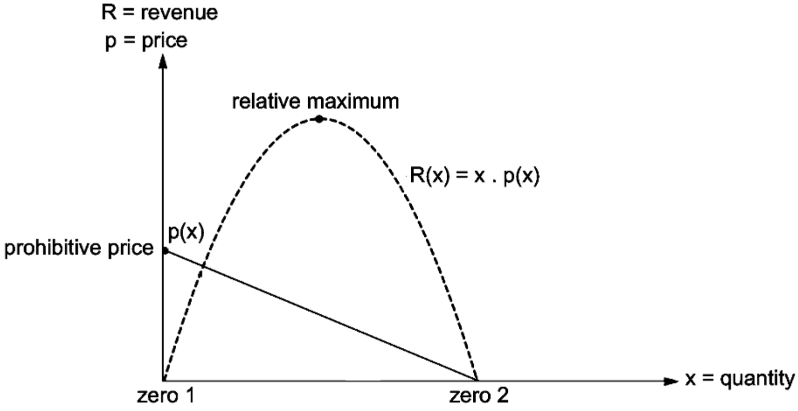


Fig. 13.5: Behaviour of Revenues Based on a Quadratic Revenue Function

Example 1:

A manufacturer offers a product at a fixed price of \$5/QU; $p = \$5/\text{QU}$. In a given period, 300 units of this product are sold; $x = 300$ QU. The revenue of this company with an output quantity of 300 QU is therefore $R(x) = \$5/\text{QU} \cdot 300 \text{ QU} = \$1{,}500$.

Example 2:

An inverse demand function is given as $p(x) = -2x + 20$. In order to determine the revenue function, the inverse demand function must be multiplied by x. The revenue function is calculated as follows:
$R(x) = (-2x + 20) \cdot x = -2x^2 + 20x$.

The two zeros of the revenue function are calculated below:

$$R(x) = 0$$

$$-2x^2 + 20x = 0$$

$$x^2 - 10x = 0$$

$$x_1 = -\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$x_1 = \frac{10}{2} + \sqrt{\left(-\frac{10}{2}\right)^2 - 0}$$

$$x_1 = 5 + \sqrt{25 - 0}$$

$$x_1 = 5 + 5 = 10$$

The first zero is at $(10|0)$.

$$R(x) = 0$$

$$x_2 = -\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$x_2 = \frac{10}{2} - \sqrt{\left(-\frac{10}{2}\right)^2 - 0}$$

$$x_2 = 5 - \sqrt{25 - 0}$$

$$x_2 = 5 - 5$$

$$x_2 = 0$$

The second zero is at $(0|0)$.

With the quantities produced of 0 QU and 10 QU, there is no revenue generated.

Test: The two zeros of a revenue function that are dependent on the price are $x = 0$ and $x = \frac{b}{m}$ (see above). In this case, the zeros are at $x = 0$ QU and at $x = \frac{20}{2} = 10$ QU, which corresponds to the solution shown above. The saturation quantity is 10 QU.

To determine the maximum of the revenue function, the derivation of this function is required. The first derivative of the revenue function $R(x) = -2x^2 + 20x$ is $\frac{dR(x)}{dx} = -4x + 20$. Following this, the *marginal revenue function* is set to zero:

$$\frac{dR(x)}{dx} = 0$$

$$-4x + 20 = 0$$

$$4x = 20$$

$$x = 5$$

$$\frac{d^2R(x)}{dx^2} = -4 < 0$$

The maximum of the revenue is located at $x = 5$ QU at a price of $p(5) = -2 \cdot 5 + 20 = \$10$/QU. The maximal revenue that can be achieved is $R(5) = 10 \cdot 5 = 50$ QU.

Test: The maximal revenue (peak of a revenue function which is dependent on the price) is located at the point with the coordinates $x = \frac{b}{2m}$ and $R\left(\frac{b}{(2m)}\right) = \frac{-b^2}{4m} + \frac{b^2}{2m}$ (see above):

$x = \frac{20}{(2 \cdot 2)} = 5$ QU;
$R\left(\frac{b}{(2m)}\right) = R\left(\frac{20}{4}\right) = \frac{-20^2}{4 \cdot 2} + \frac{20^2}{2 \cdot 2} = -50 + 100 = 50$ QU.

13.8 Cost Functions

The *total cost function* shows the relationship between the total production costs within a particular process or for manufacturing a product C and the quantity produced or (externally) purchased x (Fig. 13.6). The total cost function usually contains a fixed cost portion, the *fixed total costs* C_f and a variable cost portion, the *variable total costs* $C_v(x)$:

$$C(x) = C_f + C_v(x) \text{ with } x = \text{produced or (externally) purchased}$$

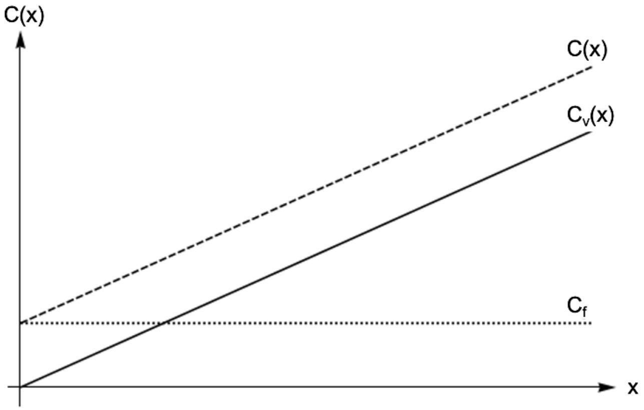


Fig. 13.6: Behaviour of the Total Costs Based on a Linear Cost Function

The *unit costs* $c(x)$ can be calculated with $c(x) = \dfrac{C(x)}{x}$ (Fig. 13.7).

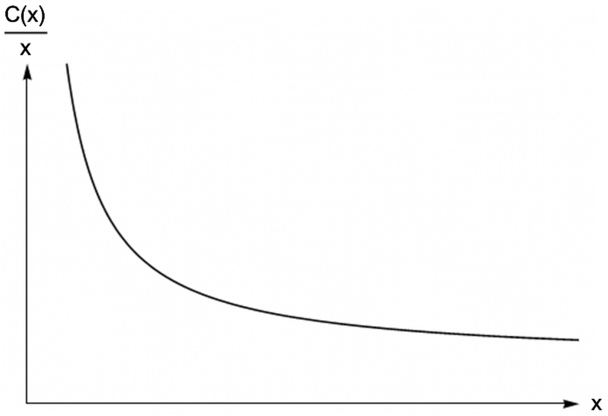


Fig. 13.7: Behaviour of the Unit Costs Based on a Linear Cost Function

The **fixed costs** C_f are *independent of any business activities* (Fig. 13.8), i.e. of production or quantity produced. The fixed costs remain unaffected by any change in the quantity produced. Fixed costs are, for example, warehouse rental costs.

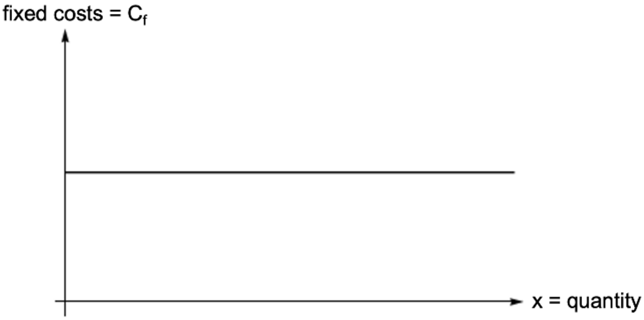


Fig. 13.8: Behaviour of the Fixed Costs Based on a Linear Cost Function

If the fixed costs are distributed linearly over the produced or purchased quantity x, the result is the *fixed unit costs* $\frac{C_f}{x}$. The fixed unit costs decrease as the quantity increases (Fig. 13.9). There is a so-called *fixed cost degression*, since the fixed costs per unit decrease (degressively) with each unit that is additionally produced/purchased.

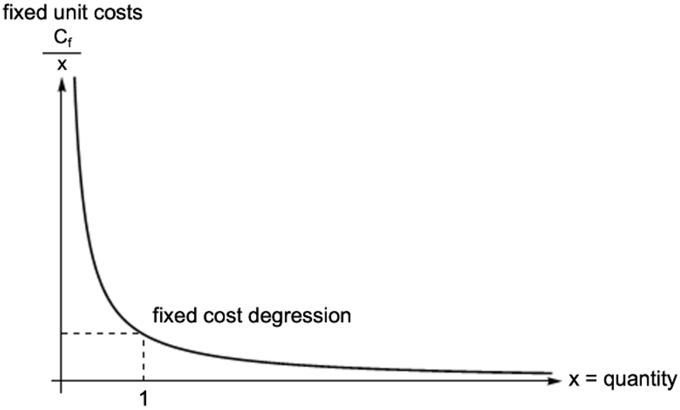


Fig. 13.9: Behaviour of the Fixed Unit Costs Based on a Linear Cost Function

The **variable costs** $C_v(x)$ cover the portion of the total cost function that varies with the quantity produced (Fig. 13.10). Examples of variable costs are commissions or the material or energy costs incurred for production. According to the *costs-by-cause principle*, variable costs can be assigned to every single unit that is produced/purchased in form of *variable unit costs* $c_v(x)$ with $c_v(x) = \frac{C_v(x)}{x}$. For a linear cost function, the variable unit costs correspond to the *marginal cost* $\frac{dC(x)}{dx} = c_v(x)$ and are constant (Fig. 13.11).

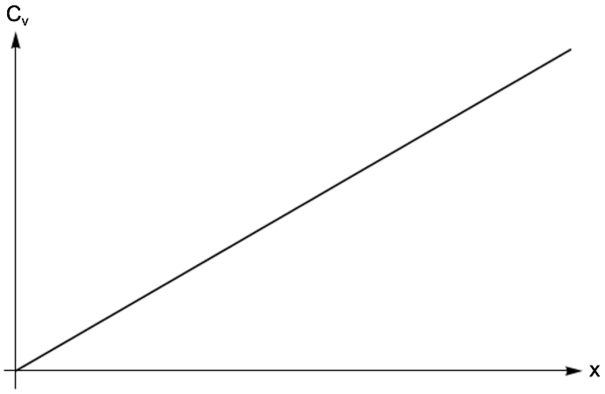


Fig. 13.10: Behaviour of the Variable Costs Based on a Linear Cost Function

Fig. 13.11: Behaviour of the Variable Unit Costs Based on a Linear Cost Function

The *marginal cost function* is the first derivative of the total cost function:

$$\frac{dC(x)}{dx} = C'(x) \qquad .$$

The marginal cost function describes the slope of the total cost function in dependence of the quantity x (Fig. 13.12). The *marginal cost* $x = x_0$ measures the expense incurred in producing or procuring one additional unit of that good, or the reduction in total costs that results from reducing the production or procurement of that good by one unit, $x = x_0$.

In a linear cost function, the behaviour of the marginal costs corresponds exactly to that of the variable unit costs.

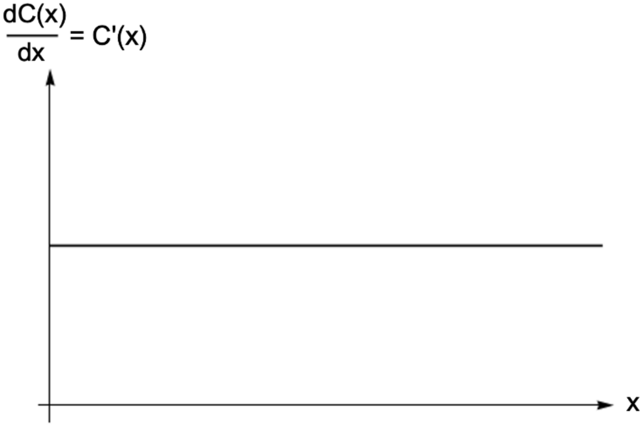


Fig. 13.12: Behaviour of the Marginal Costs Based on a Linear Cost Function

If the total cost function is not linear, it can also behave alternately, as shown graphically in Fig. 13.13. The mathematical principles of alternative cost trends are given in Tab. 13.1.

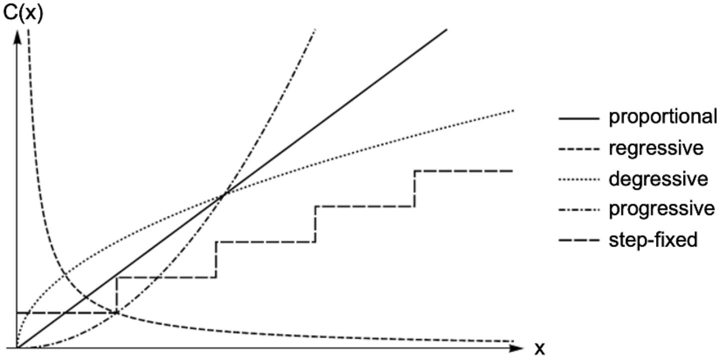


Fig. 13.13: Alternative Cost Trends

- **Degressive cost trend**: As the production/procurement increases, the total cost $C(x)$ has a disproportionately slow increase with a growing quantity x. For example, if the quantity produced increases by $5\,\%$, the production costs increase by only $3\,\%$. A degressive tendency of the total costs is known, for example, in the (continuous) offering of discounts. The procurement costs per unit (unit costs) decrease with the increasing quantity procured.

- **Progressive cost trend**: As the production/procurement increases, the total cost $C(x)$ has a disproportionately fast increase with a growing quantity x. For example, if the quantity produced increases by $5\,\%$, the production costs increase by $7\,\%$. Labour costs can rise progressively if, for instance, overtime is paid at a rate that increases disproportionately over time. The labour costs per unit (*unit labour costs*) then rise progressively with an increasing amount of overtime hours.

- **Regressive cost trend**: As the production/procurement increases, the total cost $C(x)$ has a disproportionately slow decrease with a growing quantity x.

- **Step-fixed cost trend**: As the production/procurement increases, the fixed costs $C_f(x)$ included in the total costs $C(x)$ grows rapidly (step-fixed) with an increasing quantity x at a certain position $x = x_0$. This can occur during the course of development to the same or a different extent. A sharp (step-fixed) decrease of the fixed costs at a certain position $x = x_0$ is also possible.

Behaviour	$C(x)$	Example	Marginal costs $dC(x)/dx$	Unit costs $c(x)$
proportional	bx	x	b	b
degressive	$x^{\frac{b}{d}}$ with $b < d$	$\sqrt{x}$	$\frac{b}{d}x^{\frac{b}{d}-1}$	$x^{\frac{b}{d}-1}$
progressive	bx^d	x^2	dbx^{d-1}	bx^{d-1}
regressive	bx^{-d}	$\frac{1}{x}$	$(-d)bx^{-d-1}$	bx^{-d-1}
fixed	a	100	0	$\frac{a}{x}$
step-fixed	*exemplary see example*	$\begin{cases} 100 \text{ for } x < 10 \\ 250 \text{ for} \\ 10 \leq x < 20 \\ 500 \text{ for } x \geq 20 \end{cases}$	*zero in every interval; not differentiable at the jump discontinuities*	*the following applies to the example:* $\begin{cases} \frac{100}{x} \text{ for } x < 10 \\ \frac{250}{x} \text{ for} \\ 10 \leq x < 20 \\ \frac{500}{x} \text{ for } x \geq 20 \end{cases}$

Table 13.1: Alternative Cost Trends with $a \in \mathbb{R} > 0;\ \ b \in \mathbb{R} > 0;$ $d \in \mathbb{N} > 1$

Example 1:

A company manufactures stuffed toys. The material costs to produce a stuffed toy are $\$6$/QU. The rental costs for the operation of the facility including administration are $\$300$ per month. In a considered period, 80 stuffed toys are made. The total costs with fixed costs of $\$300$ and variable costs of $\$6$/QU are therefore:

$$C(x) = C_f + C_v(x) = \$300 + \$6/\text{QU} \cdot 80\ \text{QU} = \$780$$

Example 2:

The costs incurred every day of a manufacturing company are as listed: $\$1,462.50$ at 375 QU and $\$2,400$ at $1,000$ QU. In order to identify the total cost function, it is necessary to first determine how high the variable costs are. The variable costs represent the non-constant part of the total costs and can therefore be measured between the two different production levels by the difference in production costs:

$$\$2,400 - \$1,462.50 = \$937.50$$

A change in quantity of $1,000$ QU $-$ 375 QU $=$ 625 QU results in an increase in costs of $\$937.50$. The (average) variable unit costs, which correspond to the variable unit costs ($=$ marginal costs) in a linear cost function, are obtained by assigning the change in cost to the change in quantity (linear).

$$\frac{\$937.50}{625\ \text{QU}} = \$1.50/\text{QU}$$

If the variable unit costs are multiplied by the quantity produced in one of the two scenarios, the variable costs of this output quantity are calculated as follows:

$$\$1.50/\text{QU} \cdot 375\ \text{QU} = \$562.50$$

From a total $\$1,462.50$ of the production costs, $\$562.50$ comprises the variable (total) costs. Therefore, the fixed costs are: $\$1,462.50 - \$562.50 = \$900$. In other words, the total cost function is:

$$C(x) = 900 + 1.5x$$

As a test, the alternative production level should again be analyzed in the same way:

$$\$1.50\ /\text{QU} \cdot 1,000\ \text{QU} = \$1,500$$

From a total $\$2,400$, $\$1,500$ comprises the variable (total) costs.

The fixed costs are therefore: $\$2,400 - \$1,500 = \$900$.

13.9 Neoclassical Cost Function

A neoclassical cost function is a cost function of 2nd degree and is characterized by a disproportionate growth of the total costs $C(x)$ while the quantity produced x increases. The function is convex and strictly monotonically increasing (Fig. 13.14).

A distinction is also made between fixed and variable costs in the neoclassical cost function. The variable costs $C_v(x)$ increase disproportionately (progressive; convex) with an increasing quantity x. The fixed costs C_f remain unchanged (constant) at any production level.

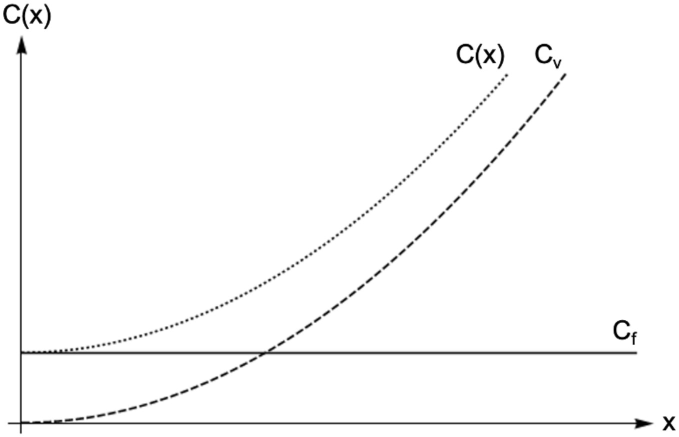


Fig. 13.14: Behaviour of the Total Costs Based on a Neoclassical Cost Function

The marginal cost function $\frac{dC}{dx} = C'(x)$ which can be calculated based on the first derivative of the total cost function $C(x)$ forming a straight line from the origin. This straight line is proportionally increasing. The second derivative of the neoclassical cost function $\frac{d^2C}{dx^2} = C''(x)$ also forms a straight line, but parallel to the x-axis (Fig. 13.15).

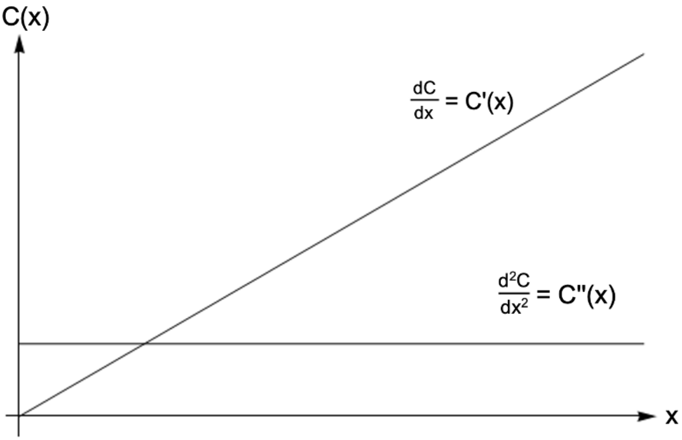


Fig. 13.15: The First and Second Derivatives of a Neoclassical Cost Function

The fixed unit costs $\frac{C_f}{x}$ are reduced with every unit that is additionally produced (fixed cost degression). Meanwhile, the variable unit costs $\frac{C_v(x)}{x}$ increase strictly monotonously (Fig. 13.16).

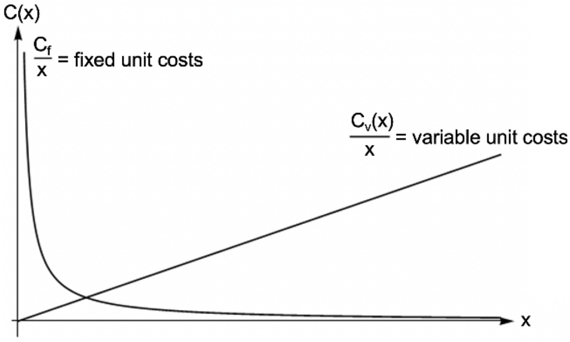


Fig. 13.16: Behaviour of the Variable and Fixed Unit Costs Based on a Neoclassical Cost Function

The total unit cost $\frac{C(x)}{x}$ is at first degressively decreasing and then progressively increasing once the (relative) minimum is reached. Since the increasing rate of the variable unit costs is stronger than the decreasing rate of the fixed unit costs, the total unit cost function $\frac{C(x)}{x}$ is increasing after the (relative) minimum, as graphically shown in Fig. 13.17.

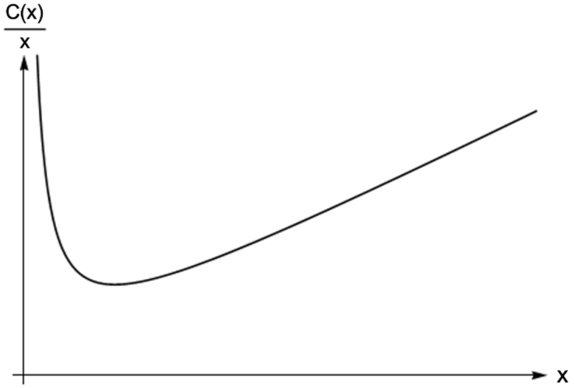


Fig. 13.17: Behaviour of the Total Unit Costs Based on a Neoclassical Cost Function

The *operational optimum* describes the output quantity x, where the average total costs (unit costs) $\frac{C(x)}{x}$ are minimal. The corresponding unit price $/QU determines the *long-term lower price limit* since this price, according to the *full-cost accounting* principle, should not fall below this limit (in the long term) in order to avoid losses. If the losses are permanent, (private-sector) production is not a viable option. If the *long-term lower price limit* is undercut, a *direct-cost accounting* must be performed to examine whether the price that can be realized in the market is still above the short-term lower price limit and whether it still ensures a positive *contribution margin*.

With a unit price equivalent to the operational optimum, the production of this good generates no profit/loss in the long term. It can be meaningful for a manufacturer to choose the selling price that corresponds to the price at the operational optimum if the manufacturer faces *cut-throat competition* with this product or if he wants to produce or distribute this product with no intention of making a profit.

The operational optimum can be calculated by setting the first derivative of the unit cost function, $\frac{dc(x)}{dx}$ with $c(x) = \frac{C(x)}{x}$, to zero. If the x-value calculated with this method is inserted into the unit cost function, the minimum unit cost can be determined and hence also the *long-term lower price limit*. Alternatively, the same result can be achieved by defining the intersection of the *marginal cost curve* $\frac{dC(x)}{dx} = C'(x)$ with the *unit cost curve* $c(x) = \frac{C(x)}{x}$ by equating these two functions to each other.

The developments of the total costs, unit costs and marginal costs of a neoclassical cost function are graphically shown in Fig. 13.18.

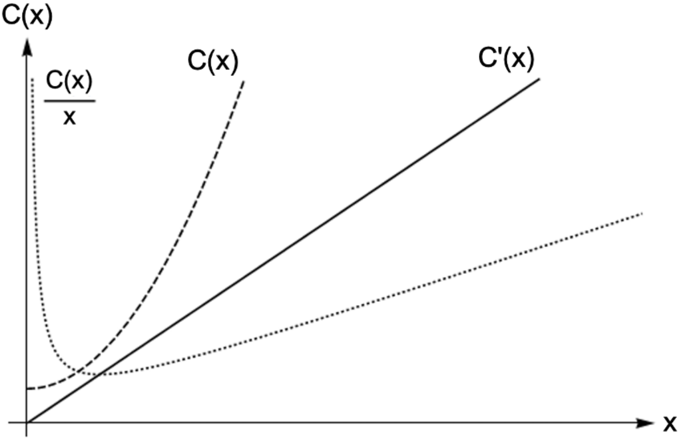


Fig. 13.18: Total Costs, Unit Costs and Marginal Costs of a Neoclassical Cost Function

Example 1:

A jewelry company produces necklaces with a manufacturing process that can be represented by the total cost function $C(x) = 10 + 0.5x^2$. Due to an increase in demand for this product, the company decides to produce more to satisfy the demand. Before the increase in demand, a total cost of $\$1,810$ per month was generated at a quantity produced of 60 QU per month. However, the demand has now increased by $20\,\%$. How much does the current production cost?

60 QU $\cdot\, 1.2 = 72$ QU

$C(72) = 10 + 0.5 \cdot 72^2 = \$2,602$

With a demand increase of $20\,\%$, the total production costs are now $\$2,602$, i.e. they have increased by $\$792$ (disproportionately).

Example 2:

A watch manufacturer thinks that the average costs he has to pay monthly seem to be relatively high. The (total) cost function for the production of his watches is $C(x) = 4,200 + x^2$.

To get a more precise overview of the average costs for each watch produced (unit costs), he divides the (total) costs by 60 QU, the average amount of watches that are produced monthly.

$$c(60) = \frac{C(60)}{60} = \frac{4,200 + 60^2}{60} = \$130/\text{QU}$$

The unit costs of 60 manufactured watches per month are therefore $\$130$/QU.

If, however, he wants to produce according to the operational optimum, the minimum of the unit cost function must be identified:

$$c(x) = \frac{C(x)}{x} = \frac{4,200 + x^2}{x} = \frac{4,200}{x} + x$$

$$\frac{dc(x)}{dx} = -4,200 \cdot x^{-2} + 1$$

$$-4,200 \cdot x^{-2} + 1 = 0$$

$$x^2 = 4,200$$

$$x = \sqrt{4,200} = 64.81 \text{ QU}$$

$$c(64.81) = \$129.61/\text{QU}$$

Test: At the operational optimum, the function value at the minimum of the average total costs corresponds to the function value of the corresponding marginal costs:

$$\frac{dC(x)}{dx} = 2 \cdot x$$

$$\frac{dC(x)}{dx}(64.8074) = 2 \cdot 64.8074 = \$129.61/\text{QU}$$

If the watch manufacturer had wanted to optimise the (average) unit costs for this month, he should have produced 64.8074, i.e. 65 watches, instead of 60. His average cost saving per watch would then have been $\$0.39$/QU ($= \$130$/QU $- \ \$129.61$/QU).

Example 3:

The finance department of ProductionX Ltd. wants to find out when the average total costs in the company are minimised. Based on a given cost function of $C(x) = 200 + 0.2x^2$, the operational optimum can be determined as shown below:

$$c(x) = \frac{C(x)}{x} = \frac{200 + 0.2x^2}{x} = \frac{200}{x} + 0.2 \cdot x$$

$$\frac{dc(x)}{dx} = -200 \cdot x^{-2} + 0.2$$

$$-200 \cdot x^{-2} + 0.2 = 0$$

$$x^2 = \frac{200}{0.2} = 1{,}000$$

$$x = \sqrt{1{,}000} = 31.62 \text{ QU}$$

$$c(31.62) = \$12.65/\text{QU}$$

Test: At the operational optimum, the function value at the minimum of the average total costs corresponds to the function value of the corresponding marginal costs:

$$\frac{dC(x)}{dx} = 0.4 \cdot x$$

$$\frac{dC(x)}{dx}(31.62) = 0.4 \cdot 31.62 = \$12.65/\text{QU}$$

With an output quantity of 31.62 QU, the average total costs of the production discussed here are minimised.

13.10 Cost Function According to the Law of Diminishing Returns

A *(total) cost function according to the law of diminishing returns* represents a cubic function. It begins at the y-intercept, which is determined by the fixed costs, and increases degressively from the y-intercept onwards. It then increases progressively from the (concave/convex) inflection point onwards. Typically, this development of the total cost curve that models diminishing returns can be explained by decreasing growth rates (degressive) at first and increasing growth rates (progressive) of the total costs after the inflection point. The spot at which the inflection point is located is also called *the point of diminishing returns*. The graph of the (total) cost function in accordance with the law of diminishing returns is first concave and then, from the inflection point, changes to convex.

The (total) cost function according to the law of diminishing returns only assumes positive values and does not contain any local (relative) extremes. The curve of this function with the general form $C(x) = ax^3 + + bx^2 + cx + d$ can be graphically represented as graphically shown in Fig. 13.19.[1]

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[1] If there are local extremes in a cubic total cost function, there is a relative maximum on the left of the inflection point and a local minimum on the right. However, the total cost function based on the law of diminishing returns actually has no local extremes by definition. It is strictly monotonically increasing, as long as the inflection point is not a saddle point.

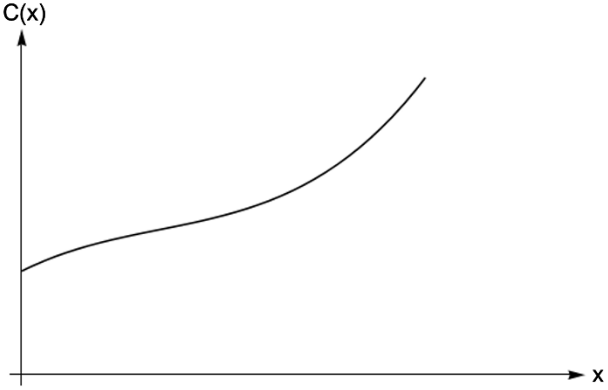

Fig. 13.19: Behaviour of the Total Costs Based on a Cost Function According to the Law of Diminishing Returns

A cost function based on the law of diminishing returns with $C(x) = ax^3 + bx^2 + cx + d$ consists of a variable part $C_v(x) = ax^3 + bx^2 + cx$ (function of the variable costs) and a fixed part $C_f = d$ (fixed costs). The variable costs in this case also first behave degressively with increasing quantity x and then progressively starting from the inflection point of $C_v(x)$. The variable cost function $C_v(x)$ starts at the origin. The fixed costs C_f are constant and form a parallel line to the x-axis and has a value of d (Fig. 13.20).

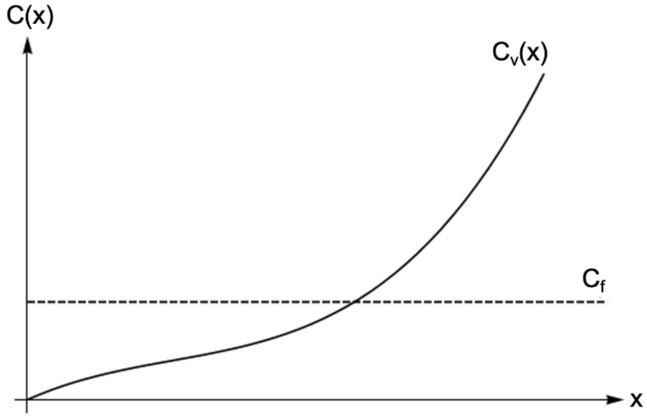

Fig. 13.20: Behaviour of the Variable and Fixed Costs Based on a Cost Function According to the Law of Diminishing Return

The *marginal cost function* $\frac{dC(x)}{dx} = C'(x)$ that associates with the (total) cost function based on the law of diminishing returns also has only positive values. In contrast to the total costs, the marginal cost function has a local minimum:

$$\frac{dC(x)}{dx} = 3ax^2 + 2bx + c \qquad \text{(marginal cost function)}$$

with

$$C(x) = ax^3 + bx^2 + cx + d \qquad \text{(total cost function)}$$

The minimum of the marginal cost function:

$$3ax^2 + 2bx + c = 0$$

First, the formula must be converted to the normal form so that, for example, by means of the p/q formula the zeros can be determined:

$$x^2+\frac{2b}{3a}x+\frac{c}{3a}=0$$

$$x_1=-\frac{p}{2}+\sqrt{\left(\frac{p}{2}\right)^2-q}\quad\text{with}\quad x^2+px+q=0$$

$$x_1=-\frac{\frac{2b}{3a}}{2}+\sqrt{\left(\frac{\frac{2b}{3a}}{2}\right)^2-\frac{c}{3a}}$$

$$x_1=-\frac{b}{3a}+\sqrt{\left(\frac{b}{3a}\right)^2-\frac{c}{3a}}$$

A zero is at $P_1\left(-\frac{b}{3a}+\sqrt{\left(\frac{b}{3a}\right)^2-\frac{c}{3a}}\;\middle|\;0\right)$.

$$x_2=-\frac{p}{2}-\sqrt{\left(\frac{p}{2}\right)^2-q}\quad\text{with}\quad x^2+px+q=0$$

$$x_2=-\frac{\frac{2b}{3a}}{2}-\sqrt{\left(\frac{\frac{2b}{3a}}{2}\right)^2-\frac{c}{3a}}$$

$$x_2=-\frac{b}{3a}-\sqrt{\left(\frac{b}{3a}\right)^2-\frac{c}{3a}}$$

The second zero is at $P_2\left(-\frac{b}{3a}-\sqrt{\left(\frac{b}{3a}\right)^2-\frac{c}{3a}}\;\middle|\;0\right)$.

The behaviour of the marginal cost function $C'(x)$ changes from degressively decreasing before the minimum to progressively increasing after the minimum (Fig. 13.21). It corresponds to a parabola that opens upwards. The total cost function $C(x)$ has a inflection point at the position where the marginal cost function $C'(x)$ has a minimum.

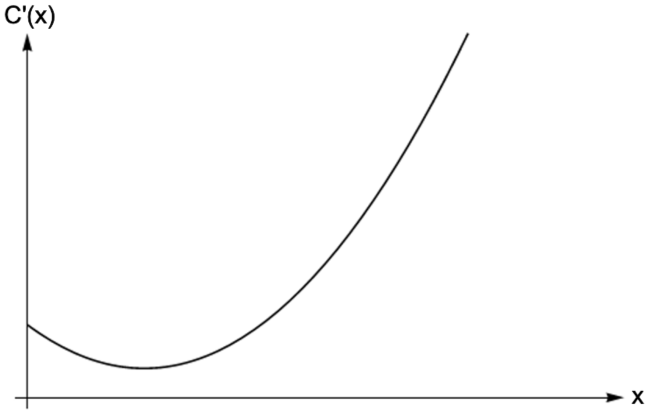

Fig. 13.21: Behaviour of the Marginal Costs Based on a Cost Function According to the Law of Diminishing Return

In the case of the total unit costs $\frac{C(x)}{x} = ax^2 + bx + c + \frac{d}{x}$, the cost trend is degressively decreasing until it reaches the (relative) minimum and increases then progressively (Fig. 13.22).

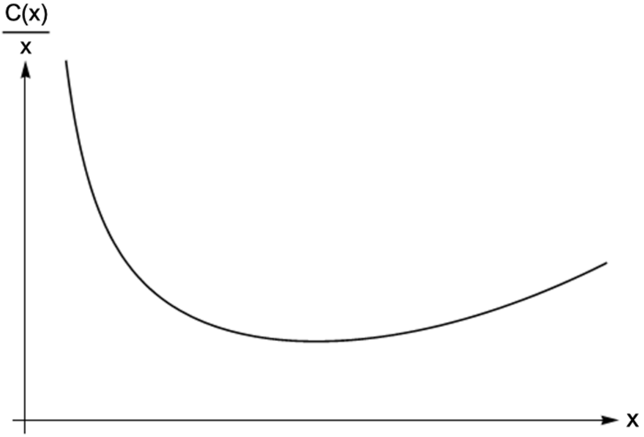

Fig. 13.22: Behaviour of the Unit Costs Based on a Cost Function According to the Law of Diminishing Return

At the *operational optimum*, the average total costs (unit costs) $\frac{C(x)}{x} = c(x)$ are minimal:

$$\frac{C(x)}{x} = c(x) = \frac{\left(ax^3 + bx^2 + cx + d\right)}{x}$$

Necessary condition: $\frac{dc(x)}{dx} = c'(x) = 0$

Sufficient condition: $\frac{d^2c(x)}{dx^2} = c''(x) > 0$

If the x-value, at which the unit costs are minimal, is inserted into the unit cost function $c(x)$, the value of the minimum unit costs (y-value) determines the *long-term lower price limit* in $/QU. This lower price limit should not be undercut in the long term, otherwise losses would be sustained. If a company is at the operational optimum, the entire unit costs including fixed costs are covered. At the operational optimum, the marginal costs of the total cost function are equal to the unit costs:

$\frac{dC(x)}{dx} = c(x)$. The corresponding x-value determines the operational optimum.

The *operational minimum* describes the output quantity x, where the variable unit costs $\frac{C_v(x)}{x} = c_v(x)$ are minimal:

$$c_v(x) = \frac{\left(ax^3 + bx^2 + cx\right)}{x}$$

Necessary condition: $\frac{dc_v(x)}{dx} = c_v{}'(x) = 0$

Sufficient condition: $\frac{d^2c_v(x)}{dx^2} = c_v{}''(x) > 0$

The corresponding value of the variable unit costs (y-value) determines the *short-term* or *absolute lower price limit* (Fig. 13.23). At the operational minimum, only the total variable unit costs are covered, not the fixed costs. The x-value of the operational minimum can be determined alternatively by equating the marginal costs of the total cost function with the variable unit costs: $\frac{dC(x)}{dx} = c_v(x)$.

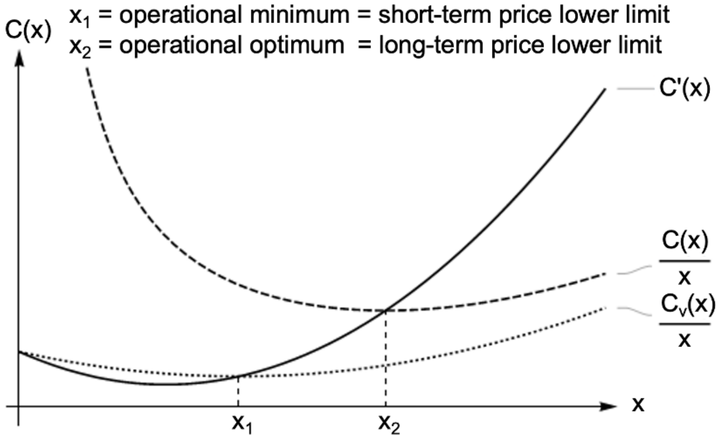

Fig. 13.23: Presentation of the Short-term and Long-term Price Lower Limit

Example 1:

A production company has a cost function in accordance with the law of diminishing returns $C(x) = 0.3x^3 - 2x^2 + 7x + 16$. The head of department wants to find out when exactly the point of diminishing returns occurs, i.e. the point at which the graph changes from degressively increasing to progressively increasing.

To determine this, the total cost function must be derived three times:

$$C(x) = 0.3x^3 - 2x^2 + 7x + 16$$

$$C'(x) = 0.9x^2 - 4x + 7$$

$$C''(x) = 1.8x - 4$$

$$C'''(x) = 1.8$$

To identify the inflection point (point of diminishing returns), the second derivative must be set to zero (necessary condition):

$$C''(x) = 0$$

$$1.8x - 4 = 0$$

$$1.8x = 4$$

$$x = \frac{20}{9} \text{ QU}$$

This x-value must now be inserted into the 3rd derivative as a check whether the sufficient condition that the 3rd derivative is not equal to zero is fulfilled:

$$C'''\left(\frac{20}{9}\right) = 1.8 > 0$$

If the 1st derivative $x = \frac{20}{9}$ QU is equal to zero, it is a saddle point. This is not the case here because:

$$C'\left(\frac{20}{9}\right) = 0.9 \cdot \left(\frac{20}{9}\right)^2 - 4 \cdot \left(\frac{20}{9}\right) + 7 = \frac{23}{9}$$

The value of the 3rd derivative at the examined position is greater than 0, i.e. at the position $x = \frac{20}{9}$ QU, there is a concave/convex inflection point (point of diminishing returns).

Example 2:

A production company wants to find out when the short-term lower price limit (operational minimum) and the long-term lower price limit (operational optimum) are reached. The production can be represented by the cost function in terms of the law of diminishing returns:
$C(x) = x^3 - 6x^2 + 60x + 100$.

To determine the *operational optimum*, the unit cost function $c(x)$ is required:

$$c(x) = \frac{C(x)}{x} = \frac{\left(x^3 - 6x^2 + 60x + 100\right)}{x} = x^2 - 6x + 60 + \frac{100}{x}$$

The x-value at the operational optimum is then determined by the minimum of the total unit cost function:

necessary condition: $c'(x) = 0$

$$2x - 6 - \frac{100}{x^2} = 0$$

The left term of this equation can be converted to a standard form by extending the equation on both sides with x^2:

$$2x^3 - 6x^2 - 100 = 0$$

Since this is a cubic function, it requires e.g. a polynomial division. The first zero is $x_1 = 5$.

$$\left(2x^3 - 6x^2 - 100\right) : (x - 5) = 2x^2 + 4x + 20 \qquad .$$

Afterwards, for instance, the p/q formula can be implemented:

$$x_2 = -\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$2x^2 + 4x + 20 = 0$$

$$x^2 + 2x + 10 = 0$$

$$x_2 = -\frac{2}{2} + \sqrt{\left(\frac{2}{2}\right)^2 - 10}$$

$$x_2 = -1 + \sqrt{(1)^2 - 10}$$

There are no solutions for x_2 and x_3 because there is a negative value in the radicand in both cases.

The sufficient condition for a minimum of the unit cost function at $x_1 = 5$ is $\frac{d^2c(x)}{dx^2} = c''(x) > 0$

$$c''(x) = 2 + \frac{200}{x^3}$$

$$c''(5) = 2 + \frac{200}{5^3} = 3.6 > 0$$

The *operational optimum* is found at $x = 5$.

Alternatively, the operational optimum can also be identified by equating the *marginal costs* $C'(x)$ to the *unit costs* $c(x)$:

$$C'(x) = c(x)$$

$$3x^3 - 12x + 60 = x^2 - 6x + 60 + \frac{100}{x}$$

$$2x^2 - 6x = \frac{100}{x}$$

$$2x^3 - 6x^2 - 100 = 0$$

The calculation procedure is the same as above. There is a zero at $x = 5$ QU.

To determine the *operational minimum*, the variable unit cost function $c_v(x)$ is required:

$$c_v(x) = \frac{C_v(x)}{x} = \frac{x^3 - 6x^2 + 60x}{x} = x^2 - 6x + 60$$

The x-value of the operational minimum is determined by the minimum variable unit costs:

Necessary condition: $c_v{}'(x) = 0 \;\Rightarrow\; 2x - 6 = 0 \;\Leftrightarrow\; 2x = 6$

$$x = 3 \text{ QU}$$

Sufficient condition: $c_v{}''(x) > 0$

$$c_v{}''(x) = 2 > 0$$

The necessary condition is thus fulfilled and the operational minimum corresponds to the quantity produced $x = 3$ QU.

Similarly, the operational minimum can also be identified by equating the marginal costs of the total cost function with the variable unit costs.

$$C'(x) = c_v(x)$$

$$3x^2 - 12x + 60 = x^2 - 6x + 60$$

$$2x^2 - 6x = 0$$

$$x^2 - 3x = 0$$

$$x_1 = -\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$x_1 = -\left(\frac{-3}{2}\right) + \sqrt{\left(\frac{-3}{2}\right)^2 - 0}$$

$$x_1 = 1.5 + \sqrt{2.25 - 0}$$

$$x_1 = 1.5 + 1.5$$

$$x_1 = 3 \text{ QU}$$

$$x_2 = -\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$x_2 = -\left(\frac{-3}{2}\right) - \sqrt{\left(\frac{-3}{2}\right)^2 - 0}$$

$$x_2 = 1.5 - \sqrt{2.25 - 0}$$

$$x_2 = 1.5 - 1.5$$

$$x_2 = 0 \text{ QU}$$

The operational minimum describes the output quantity x where the variable unit costs $\frac{C_v(x)}{x} = c_v(x)$ are minimal:

$$c_v(x) = x^2 - 6x + 60$$

$$c_v(3) = 3^2 - 18 + 60 = \$51/\text{QU}$$

$$c_v(0) = 0^2 - 0 + 60 = \$60/\text{QU}$$

The operational minimum is at 3 QU.

13.11 Direct Costs versus Indirect Costs

In cost accounting, a distinction is made not only between fixed and variable costs, but also between *direct costs* and *indirect costs*.

Direct costs can be:

- *Direct manufacturing costs*: non material-related manufacturing costs that arise directly during the production process and can be directly allocated to the manufactured product (cost object). Examples of the direct manufacturing costs are labour costs that can be directly assigned to the cost object (direct labour costs per unit, piecework wages), as well as machine costs (machine unit costs) or construction costs.

- *Direct material costs*: material costs that are included in the product that is to be manufactured and can be directly allocated to the product that is manufactured (cost object). Examples of direct material costs are raw materials, auxiliary materials, purchased parts or (preliminary) products.

- *Special direct costs*: special direct costs that are related to the production and distribution. They cannot be directly allocated to a product or (sales-related) service, but to a specific contract or project, for instance. *Special direct manufacturing costs* are, for example, costs for licenses or for special tools required for this (particular) production. *Special direct distribution costs* can be, for example, (special) packaging costs or commission costs.

Direct costs are characterized by their direct and clear allocation to a *cost object*, e.g. a product or service, or to a *cost centre*, e.g. a department, a plant or a company. Accordingly, direct costs are differentiated between *direct costs of cost object* or *direct costs of cost centre*.

In contrast to direct costs, *indirect costs* cannot be directly assigned to the reference object. Indirect costs are incurred for multiple end products or orders. They are also called *overhead costs*. Similar to direct costs, indirect costs can be divided into *indirect costs of cost object* and *indirect costs of cost centre*.

The following differences in definition are also significant for indirect costs:

- *False indirect costs / false overheads*: indirect costs that could theoretically be allocated directly as direct costs to the cost objects or cost centres, but due to reasons of economic efficiency, are assigned proportionately using classification keys (*classification based on indirect costs*), e.g. incidentals, electricity costs, lubricant costs.

- *Primary indirect costs / primary overheads*: indirect costs that are generated in the cost centres or cost centre areas themselves due to external acquisitions, i.e. resources or services (e.g. material, personnel, external services) are purchased on the external market. Since the corresponding market prices are known, the primary indirect costs can be clearly identified; there is no problem with value assessment. Consumption documentation, purchase invoices or account statements allow the primary indirect costs to be measured accurately and assigned directly to the cost centres or responsible cost centre areas.

- *Secondary indirect costs / secondary overheads*: While the primary indirect costs are incurred by resources or services outside the company, secondary indirect costs arise from internal service relationships (for example, services provided by the company health insurance funds). Secondary indirect costs are first determined during internal cost allocation; only then can they be assigned to cost centres or cost centre areas in monetary form. Usually, market prices do not apply to internal activities, so therefore, internal transfer prices must be calculated.

- *Indirect manufacturing costs / manufacturing overheads*: that part of production costs which cannot be directly allocated. The *production costs* are composed of *direct manufacturing costs* (e.g. direct labour costs, direct material costs), *indirect manufacturing costs* (e.g. salaries for supervisor and technical staff, auxiliary labour costs, costs of auxiliary materials and supplies used in production, electricity costs, imputed depreciation or imputed interest) and *special direct manufacturing costs* (e.g. special tools, construction plans, patents, licenses). Normally, indirect manufacturing costs cannot be allocated to a single unit produced, but rather to an order or a lot. In the full-cost accounting, the direct manufacturing costs usually form the reference value for the allocation of the indirect manufacturing costs (cost-plus pricing; *cost-plus indirect manufacturing costs*).

- *Indirect material costs / material overheads*: that part of the material costs that cannot be directly allocated to specific cost objects (products) in the production (e.g. procurement costs, appraisal costs, collective warehousing costs, personnel costs for employees in warehouses or in the purchasing process, depreciation for collective warehouses, i.e. warehouses where other materials are also stored). Indirect material costs are taken into account in (annual) cost center planning. In full-cost accounting, the direct material costs usually form the reference value for the allocation of the indirect material costs (cost-plus pricing; *cost-plus indirect material costs*).

- *Indirect administrative costs / administrative overheads*: that part of the administrative costs which is incurred in the administration of a company but cannot be allocated to specific products (cost objects) (e.g. salaries for the board of managers, salaries of administrative staff, office supplies, depreciation on business equipment). The indirect administrative costs can be determined within the framework of *cost centre accounting as part of the cost distribution sheet*. In the *unit-based cost unit accounting*, the *manufacturing costs per* *unit produced* forms the reference value for the allocation of the indirect material costs; in the *time-based cost unit accounting*, the *manufacturing costs per unit sold*, i.e. the revenue during the observed period (usually annually) forms the reference value for the allocation of the indirect material costs (cost-plus pricing; *cost-plus indirect administrative costs*).

There are various *cost allocation principles* for assigning costs to the corresponding reference values. A distinction is made between *one-dimensional* and *multi-dimensional cost allocation principles*:

13.11.1 One-Dimensional Cost Allocation Principles

The (only) reference value here is the activity, i.e. in the case of a (total) cost function $C(x)$ it is the quantity x which is produced or performed.

Principle of Causation

Only the costs that are additionally incurred in the process of producing this (one) unit can be allocated to the cost object. The costs assigned to a (single) unit of the cost object therefore correspond to their marginal costs. According to the causation principle, the *direct costs* as well as the *variable activity-based indirect costs* are allocated to the cost object. An allocation of the *fixed activity-based indirect costs* is not possible. The causation principle is applied in a German costing methodology, *Grenzplankostenrechung*, translated as either *marginal planned cost accounting* or *flexible analytic cost planning and accounting*.

Principle of Utilisation

The costs that are additionally incurred in the process of producing this (one) unit can be allocated to the cost object. The costs assigned to a unit of the cost object thus correspond to their marginal costs and their *used capacity costs*. The used capacity costs comprise the part of the fixed costs that is allocated to the capacity under usage. The unused part of the fixed costs is called *idle capacity costs*. The utilisation principle is applied in the *activity-based accounting* (ABC).

Principle of Averages

In addition to direct costs, the *average indirect costs* are allocated to the cost object. The average indirect costs are calculated by distributing the total indirect costs linearly by the means of division to the units x which are produced. If the costs cannot be allocated to the cost objects according to the *principle of causation* in *full-cost accounting*, they can be distributed according to the principle of averages. The variable costs, which are activity-independent costs, are divided by the number of units produced x and distributed linearly to the units produced. In the case of a *single-product company*, the total costs $C(x)$ are divided by the total number of units produced x. However, in the case of a *multi-product company*, the distribution of the indirect costs to the cost objects must be carried out by using a *classification based on indirect costs*.

Principle of Plausibility

Those costs that are (plausibly) associated with a (different) cost type are assigned to the cost object. For example, the allocation of the, e.g., indirect material costs, can be oriented to the direct material costs (linear) if there seems to be a plausible (linear) relationship between the direct material costs and the associated indirect material costs. Usually, such an allocation is linear, but a nonlinear relationship between direct costs and indirect costs can also be plausible.

Principle of Financial Viability

The (pro rata) costs are allocated to the cost object according to its (pro rata) sales revenue. High-revenue (low-revenue) products should be charged with a higher (lower) proportion of the indirect costs or of the total costs (direct and indirect costs). Instead of the sales revenue, either the price or the contribution margin can be chosen for the (proportional) allocation of indirect or total costs.

13.11.2 Multi-Dimensional Cost Allocation Principles

Principle of Decision

According to the principle of decision, costs and revenues can only be clearly allocated to each other or to another reference object if they are related to the same (business or operational) decision as the reference object itself. According to this principle, costs are only allocated to a product or service if they are directly caused by the decision to manufacture this product or provide this service.[2]

Principle of Identity

The principle of identity is a further development of the principle of decision. It forms the cost-accounting basis for the *relative calculation of direct costs*. According to the principle of identity, a business decision is based on three dimensions:[3]

- Performance | What is to be produced or performed in what quantities?
- Organisation | Who is supposed to produce the product(s) or perform the service(s)?
- Time | When or to what point in time is this to be produced or performed?

In contrast to the one-dimensional cost allocation principles, the reference objects here are defined in three dimensions, e.g. in the production of goods in the dimensions:

- Operational performance (product unit, product type, product group, product line),

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[2] Cf. Riebel, P. (2013): Einzelkosten- und Deckungsbeitragsrechnung. Grundfragen einer markt- und entscheidungsorientierten Unternehmensrechnung, 6th Edition, Wiesbaden; Schweitzer, M.; Küpper, H.-U.; Friedl, G.; Hofmann, C.; Pedell, B. (2015): Systeme der Kosten- und Erlösrechnung, 11th Edition, Munich.

[3] Cf. ibid.

- Organisational area (company, plant, group, department),
- Time period (month, quarter, year).

Within these dimensions, the hierarchies of reference objects can possibly be observed through relationships of superordination and subordination. Therefore, the direct costs at a higher decision-making level can also represent indirect costs at a lower decision-making level.

Example 1:

An employee of ToyFactoryX Ltd. manages to produce a customised toy car in 20 minutes. The costs for the required material are $\$3$/QU and the machine costs are $\$5$/QU. The gross hourly wage for this employee is $\$24$/h.

The *direct manufacturing costs* which are directly allocable to the production of a toy car can be calculated as shown below:

$20\text{min/QU min} \cdot \$24/60\text{ min} = \$8/\text{QU}$

The *direct manufacturing costs* are therefore $\$8$/QU + $\$5$/QU $=$ $\$13$/QU. The *direct material costs* are $\$3$/QU.

Example 2:

The rent to be paid per day for the production hall of ToyFactoryX Ltd. is $\$60$/day. These costs cannot be directly allocated to every toy car manufactured and therefore represent indirect costs.

The direct costs of $\$16$/QU can be directly assigned to the model produced here; direct manufacturing costs of $\$13$/QU and direct material costs of $\$3$/QU.

50 units of this toy car are made every day. According to the cost-plus pricing, the indirect costs, in this case the rental costs, should be distributed proportionally to the direct costs of this model produced here. The corresponding cost-plus ratio is calculated as shown below:

$\$60$/day: rent for the production hall (indirect costs per day)

$\$16$/QU: direct costs (direct manufacturing costs and direct material costs) per toy car x

50 QU $= \$300$/day (direct costs per day)

$$\text{Surcharge of total costs} = \frac{\text{indirect costs}}{\text{direct costs}} \cdot 100\%$$
$$= \frac{\$60}{\$300} \cdot 100\% = 20\%$$

The total manufacturing costs of these toy cars can thus be calculated ceteris paribus, i.e. without taking other costs into account. The calculation is $\$16$/QU $\cdot\ 1.2 = \$19.20$/QU.

Example 3:

The company ToyFactoryX Ltd. would now like to find out how high the total manufacturing costs are a) per toy car produced and b) per day. The direct manufacturing costs of a toy car are $\$13$/QU. The company is known for the special spray paint for toy cars. This paint is the unique selling proposition (USP) of the company. The cost of painting a toy car is $\$3$/QU. In addition, the total indirect manufacturing costs account for 30% of the direct manufacturing costs. On a working day, 50 toy cars are made.

a) Manufacturing costs for each toy car that is produced:

Direct manufacturing costs $\$13$/QU

\+ Special direct manufacturing costs 3/QU

\+ Indirect manufacturing costs $\$13$/QU $\cdot\ 0.3 = \$3.9$/QU

= Manufacturing costs $\$19.9$/QU

The manufacturing costs for each toy car that is produced are $\$19.9$/QU.

b) Manufacturing costs per day:

50 toy cars are produced daily.

$\$19.9/\text{QU} \cdot 50\ \text{QU} = \$995$

At ToyFactoryX Ltd., the manufacturing costs are $\$995$ per working day.

13.12 Profit Function

A profit function $P(x)$ represents the profit of a company, which depends on the quantity x which is produced or sold. The profit $P(x)$ can be calculated by subtracting the (total) cost $C(x)$ from the revenues $R(x)$:

$$P(x) = R(x) - C(x)$$

$$P(x) = [p(x) \cdot x] - [C_f + C_v(x)]$$

with $p =$ (selling) price and $x =$ quantity produced or sold.

If the revenues $R(x)$ exceed the total costs $C(x)$, profits are generated, $P(x) > 0$. The company is *profitable*.

However, if the total costs $C(x)$ exceed the revenues $R(x)$, negative profits, i.e. *losses*, are generated, $P(x) < 0$. The company is *unprofitable*.

The position, i.e. the x-value, at which a profit is earned for the first time, is called the *break-even point* (BEP). The break-even point is determined by the first zero of the profit function $P(x)$ and marks the beginning of the profit zone. The *profit limit* is located at the end of the profit zone. There is another zero of the profit function $P(x)$ here. At the break-even point as well as at the profit limit, the costs and the revenues coincide.

The break-even point and the profit limit can be determined by:

$$P(x) = 0 \quad \text{or} \quad C(x) = R(x)$$

At the *profit maximum*, the x-value of the profit function is where the highest possible profit is made. The profit maximum can be calculated as shown below:

necessary condition: $P'(x) = 0$

sufficient condition: $P''(x) < 0$

A distinction must be made as shown below for the profit function:

I. The price p is constant:

This means that the price is fixed, i.e. there is no causality between the selling price p and the quantity sold x, $p = constant$, $p \neq p(x)$. The *inverse demand function* is parallel to the x-axis and the *revenue function* is represented by a straight line from the origin (Fig. 13.24). See also the chapter on the revenue function.

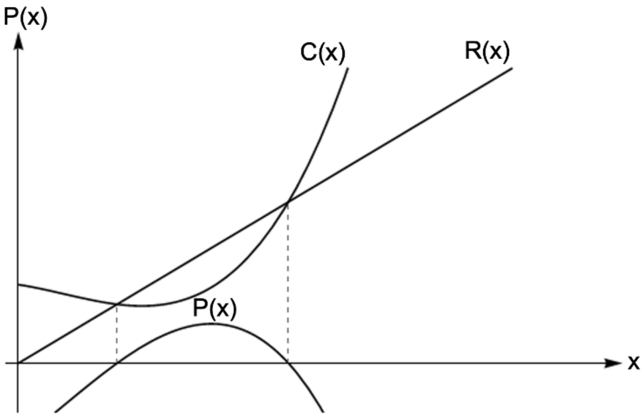

Fig. 13.24: Behaviour of Profit Based on a Linear Revenue Function

II. The price $p = p(x)$ is variable:

There is a causality between the selling price, $p = p(x)$, and the quantity sold $x = x(p)$. The *inverse demand function* $p = p(x) = -mx + b$ corresponds to a strictly monotonically decreasing straight line, of which the y-intercept b is determined by $x = 0$ (prohibitive price) and which intersects the x-axis at $p = 0$ (saturation quantity). The *profit function* is represented by a parabola opening downwards, starting at the origin, i.e. at the point $(0|0)$ and ending at the point $(x_{max}|0)$ (Fig. 13.25). x_{max} corresponds to the *saturation quantity* at a price of $p_{min} = 0$. The maximum of the profit function is at the point with the coordinates $x = \frac{b}{(2m)}$ and $R\left(\frac{b}{(2m)}\right) = -m \cdot \left(\frac{b}{(2m)}\right)^2 + b\left(\frac{b}{(2m)}\right) = \frac{-b^2}{4m} + \frac{b^2}{2m}$

See also the chapter on the profit function.

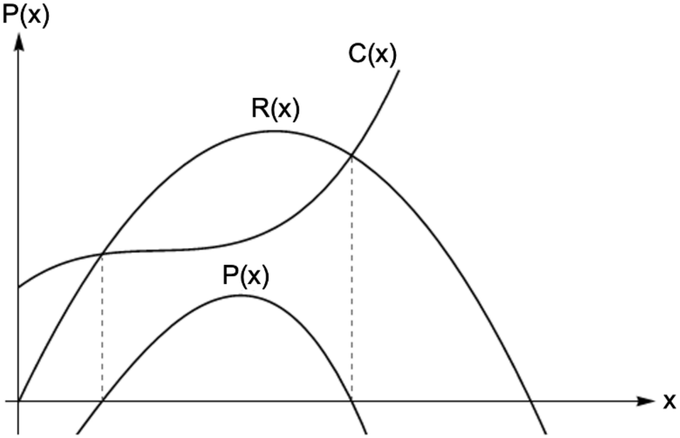

Fig. 13.25: Behaviour of Profit Based on a Quadratic Revenue Function

When a company is in a *monopolistic position*, the *Cournot point* - which is named after the French economist Antoine Augustin Cournot (1801-1877) - describes the price-quantity-combination that maximises a monopolist's profit. The Cournot point represents the monopolistic pricing. Typically, it is located on the left side of the revenue maximum. A smaller quantity of the good x is sold when the profit is maximised than when the revenue is maximised (Fig. 13.26).

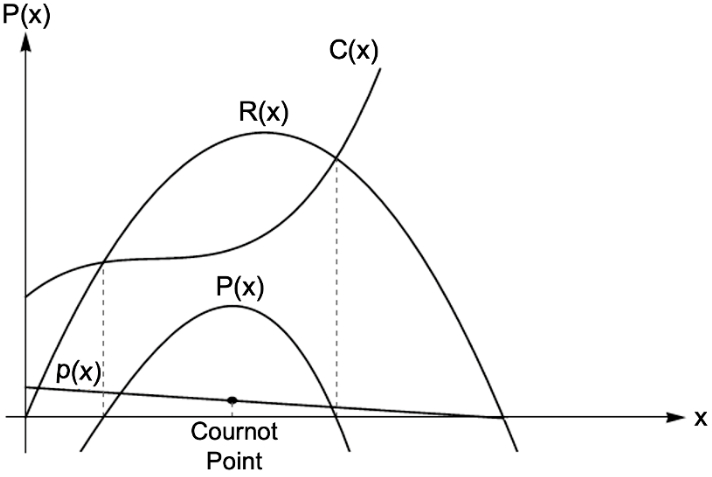

Fig. 13.26: The Cournot Point

Example 1:

The process of producing a quantity of goods x is described as shown below:

- The selling price of a good is constant and is $\$100//\text{QU}$.
- The total costs are calculated with the cost function in accordance with the law of diminishing returns: $C(x) = x^3 - 3x^2 + 52x + 50$ $[\$]$.

A manufacturer wants to find out when the firm reaches the *break-even point* and the *highest possible profit*.

To determine the *break-even point*, the *profit limit* and the *profit maximum*, the first step is to define the profit function:

$$P(x) = R(x) - C(x)$$

$$P(x) = 100x - (x^3 - 3x^2 + 52x + 50)$$

$$P(x) = -x^3 + 3x^2 + 48x - 50$$

Calculation of the *break-even point* and the *profit limit*:

$$P(x) = 0$$

$$-x^3 + 3x^2 + 48x - 50 = 0$$

As the profit function is a cubic function, a polynomial division, for example, can reveal the possible zeros. A zero is found at $x = 1$.

$$(-x^3 + 3x^2 + 48x - 50) : (x - 1) = -x^2 + 2x + 50$$

To determine other possible zeros, the p/q formula can be used:

$$x_{2,3} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

the following applies for $-x^2 + 2x + 50 = 0$,
which is also $x^2 - 2x - 50 = 0$:

$$x_{2,3} = -\left(\frac{-2}{2}\right) \pm \sqrt{\left(\frac{-2}{2}\right)^2 + 50}$$

$$x_{2,3} = 1 \pm \sqrt{(-1)^2 + 50}$$

$$x_2 = 1 + \sqrt{51} \approx 8.14$$

$$x_3 = 1 - \sqrt{51} \approx -6.14 \quad \text{(not economically relevant)}$$

If the positive zeros of the break-even point are interpreted economically, the *break-even point* is positioned at $x = 1$ QU. The *profit limit* is reached at approximately $x = 8.14$ QU.

Calculation of the *profit maximum*:

necessary condition: $P'(x) = 0$

$$P'(x) = -3x^2 + 6x + 48 = 0$$

$$x^2 - 2x - 16 = 0$$

$$x_{1,2} = -\left(\frac{-2}{2}\right) \pm \sqrt{\left(\frac{-2}{2}\right)^2 + 16}$$

$$x_1 = 1 + \sqrt{1+16}; \quad x_2 = 1 - \sqrt{1+16}$$

$$x_1 = 1 + \sqrt{17} \approx 5.12; \quad x_2 = 1 - \sqrt{17} \approx -3.12 \quad \text{(not economically relevant)}$$

sufficient condition: $P''(x) < 0$

$$P''(x) = -6x + 6 = 0$$

$$-6 \cdot \left(1 + \sqrt{17}\right) + 6 \approx -24.74 < 0$$

The *profit-maximising quantity* is approximately 5.12 QU.

The *profit maximum* is,

$P\left(1+\sqrt{17}\right) = P(5.12) = -5.12^3 + 3 \cdot 5.12^2 + 48 \cdot 5.12 - 50 \approx \$140.19.$

Example 2:

The monopolist XProducts wants to find out at what quantities the break-even point, the profit limit, the profit maximum and the Cournot point of their current production are located. The process of producing a quantity of goods x is described as shown:

- The selling price of a good is variable and is calculated according to the inverse demand function: $p(x) = -2x + 100$ $/QU.
- The total costs are calculated with the cost function in accordance with the law of diminishing returns: $C(x) = x^3 - 5x^2 + 52x + 50$ [$].

To determine the *break-even point*, the *profit limit* and the *profit maximum*, the first step is to define the profit function:

$$P(x) = R(x) - C(x)$$

$$P(x) = (-2x + 100) \cdot x - \left(x^3 - 5x^2 + 52x + 50\right)$$

$$P(x) = -x^3 + 3x^2 + 48x - 50$$

Since $P(x)$ is equal to the profit function of the first example with constant prices, the calculation of the *break-even point*, the *profit limit* and the *profit maximum* is done exactly as demonstrated in the first example (see above). The *break-even point* is at $x = 1$ [QU]. The *profit limit* is reached at circa $x = 8.14$ QU. The *profit-maximising quantity* is $1 + \sqrt{17} \approx 5.12$ QU. The *maximum profit* is:

$$P\left(1 + \sqrt{17}\right) = P(5.12) = -5.12^3 + 3 \cdot 5.12^2 + 48 \cdot 5.12 - 50 \approx \$140.19.$$

Calculation of the *Cournot point*:

The Cournot point describes the price-quantity-combination that maximises a monopolist's profit. The Cournot point represents the monopolistic pricing.

The profit-maximising quantity is at $x_{max}^{profit} = 1+\sqrt{17} \approx 5.12$ QU.
The *profit-maximising price* is calculated as shown below:

$$p_{max}^{profit} = -2\cdot\left(1+\sqrt{17}\right)+100 \approx \$89.76/\text{QU}$$

$$P_{max} = P\left(1+\sqrt{17}\right) = 2\sqrt{17^3} \approx \$140.19$$

The *Cournot point* is therefore found at:

$\left(x_{max}^{profit} \mid p_{max}^{profit}\right) = \left(1+\sqrt{17} \mid -2\left(1+\sqrt{17}\right)+100\right) \approx (5.12 \mid 89.76)$.

The x-value of the *Cournot point* is typically on the left side of the x-value at the *revenue maximum*:

$$R(x) = (-2x+100)\,x = -2x^2+100x$$

necessary condition: $R'(x) = -4x+100 = 0 \Rightarrow x = 25$

sufficient condition: $R''(25) = -4 < 0$

The *revenue-maximising quantity* is at:

$x_{max}^{revenue} = 25$ QU $> x_{max}^{profit} = 1+\sqrt{17} \approx 5.12$ QU.

A smaller quantity of the good x is sold when the profit is maximised than when the revenue is maximised.

The *revenue-maximising price* is calculated as follows:

$$p_{max}^{revenue} = -2\cdot 25+100 = \$50/\text{QU}$$

$$R_{max} = R(25) = -2\cdot 25^2+100\cdot 25 = \$1{,}250$$

Chapter 14

The Peren Theorem
The Mathematical Frame in Which We Live

Synopsis

Humans consume the natural resources of the Earth faster than the Earth is able to regenerate them. Mankind on the whole lives above its means and often at the expense of future generations. Current economic activity, with the aim of maximising monetary profits and generating quantitative growth and prosperity, cannot be continued. The *Peren Theorem* demonstrates that the consumption of natural resources within a closed system, as represented by Earth, is only possible if their consumption is able to naturally regenerate. If this balance is disturbed for too long a period, this will then result in the natural death of the planet. With an increasing global population, the per capita consumption of natural resources of all humans living on or from the Earth must be proportionately reduced.

The Current Human Lifestyle Cannot be Continued

Humans consume the natural resources of the Earth faster than the Earth is able to regenerate them. For many years now, human demand for natural resources has exceeded the Earth's capacity to regenerate these resources. According to the *Global Footprint Network*[1], the *Earth Overshoot Day* in the year 2017 took place on August 2nd of that year.[2] In the year before, it was on August 13th, 2016.

Mankind on the whole lives above its means and at the expense of future generations. The entire natural resources consumed after *Earth Overshoot Day* can no longer be replenished by the Earth in the same year concerned. If such an imbalance remains over the long term, then the natural resources of the Earth will be consumed up to the natural death of the planet.

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[1] Cf. Global Footprint Network (2017): http://www.footprintnetwork.org/, accessed 12 August 2023.

[2] Cf. ibid.

F. W. Peren, *Math for Business and Economics*,
https://doi.org/10.1007/978-3-662-68860-1_14

The *Living Planet Report 2012* of the *World Wide Fund For Nature (WWF)*[3] shows that roughly up to the year 2030 humans would require two planets in order to cover their need for natural resources if mankind continues to live as it has largely deemed proper thus far. Natural resources are wasted particularly within and for so-called highly developed economies; however, the definition of this concept is misleading because development that results in a human lifestyle that permits the Earth to bear homo sapiens only for approximately seven months per year with itself being consumed for the remaining five months of the same year – both at the expense of other organisms as well as future generations of its own species – can hardly be regarded as "highly developed." Current economic activity with the aim of maximising monetary profits and generating quantitative growth and prosperity for the companies and citizens of participating national economies already takes place today at the expense of those who either do not or are only partially able to benefit from this predatory ecological exploitation, or who consciously choose to not take part.

The Peren Theorem

In the year 2012, the author developed the *Peren Theorem*[4] in discussion with Wiltrud Terlau, director of the International Centre for Sustainable Development (IZNE)[5], and Reiner Clement, professor of economics at Bonn-Rhein-Sieg University in Sankt Augustin, Germany:

> *"If the users within a closed system employ*
> *its natural resources in such measure that its*
> *natural regeneration is exceeded over the long term,*
> *then the natural environment of this system will be*
> *completely exhausted."*

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[3] World Wide Fund For Nature - WWF (Ed.) (2017): http://wwf.panda.org/about_our_Earth/all_publications/living_planet_report_timeline/lpr_2012/, accessed 9 December 2022.

[4] Peren, F.W. (2012): The Peren Theorem, New York, unpublished manuscript; Peren, F.W. (2018): Das Peren-Theorem, in: Gadatsch, A. et al. (ed.): Nachhaltiges Wirtschaften im digitalen Zeitalter, Berlin, p. 419-424.

[5] International Centre for Sustainable Development – IZNE (2017): https://www.h-brs.de/en/izne, accessed 9 December 2022.

For a closed system stability is[6]:

$$R_T \leq R_{regen}$$

$$\begin{aligned}
\text{where} \quad R_T &= R_H + R_O \\
\text{and} \quad R_H &= \sum_{I=1}^{N} r_I = r_H N \\
\Leftrightarrow r_H &= \frac{\sum r_I}{N} = \frac{R_H}{N}
\end{aligned}$$

with

R_T = consumption of natural resources as a whole

R_{regen} = regeneration of natural resources as a whole

R_H = human consumption of natural resources

R_O = consumption of natural resources not caused by humans

r_I = individual per capita consumption of natural resources by humans

r_H = average per capita consumption of natural resources by humans

N = number of people who live on Earth or access its natural resources

I = human individuals who live on Earth or access its natural resources; $1, \ldots, N$

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[6] Stability is to be understood here as emancipatory in its meaning, i.e. if within a well-defined, temporal interval the inequation $R_T \leq R_{regen}$ is temporarily violated, then it is nevertheless valid altogether during this period. The scope and location of such a time period are to be selected in such a way that they contain the respectively current point in time and so that the strategic aim of a stable balance between consumed and regenerated natural resources is achieved not only over the longer term, but also for the benefit of those directly affected within the system taken into consideration.

Options for Securing Human Livelihood

In relation to mankind and the closed system of the Earth, this mathematical relationship implies that humans have the following options[7] in order to secure their existence on Earth:

1. Other consumers of natural resources on this planet are reduced; a practice that mankind already pursues. The habitats of animals and plants are diminished by humans with the consequence that plants and animals are decimated.

2. Mankind reduces itself until this theorem inverts into a positive balance, i.e. until terrestrial consumption caused by humans lies below the natural regeneration of the Earth over the long term.

3. Substantial numbers of mankind leave the Earth. Accordingly, these humans do not use any or hardly any terrestrial natural resources.

4. Mankind modifies the scope and quality of its consumption of natural resources so that this permits a regeneration of natural resources to the extent required. This would require substantial abandonment of the luxury that is understood by large parts of mankind today as prosperity. Individuals would then be entitled to far less natural resources on average than currently claimed and consumed on the average per capita.

5. The recourse to natural resources, i.e. the use, respectively the consumption, of water, soil, air, natural energies and/or sources of energy, plants and animals are put at a clearly higher price than the currently irrational case of disparity in relation to the true value of natural resources. Individual mobility would require a different quality and a clearly higher price. The consumption of meat would have to be more expensive and thus reduced. Global output chains would largely have to be shifted to local production because transportation would have to be priced in accordance with the demand for natural resources. Travel (over long-distances) would also have to be made substantially more expensive and limited.

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[7] The following list is by no means exhaustive.

6. Mankind substitutes natural resources in favor of synthetic materials; whereas the ecological requirements for the manufacture, transport, recycling and/or disposal of such plastics would also have to be attributed to human consumption of natural raw materials.

7. A more intensive circular economy, i.e. more efficient recycling of already used natural resources could slow down the process of exhaustion of the natural environment of the Earth. However, if increases in efficiency or technical progress results in rebound effects so that increases in efficiency mean that the consumer uses any savings obtained in order to demand more products or services which again consume (additional) natural resources, then increases in efficiency can also result in a so-called backfire, i.e. to rebound effects of more than one hundred percent.

Individual Prosperity Effects

The *Peren Theorem* mathematizes and emancipates a life cycle matter of course. Like every mathematical statement, this theorem is also logically true and thus indisputable in rational terms. If mankind within its terrestrial existence should be interested in a natural environment so that it secures a required (minimum) measure of quality of life for humans – that is certainly evaluated differently by each individual – then operational implementation of this theorem as soon as possible is imperative.

Conversely, this theorem also implies that an increasing global population[8] has to be accompanied with a proportionate reduction in the average per capita consumption of natural resources if it is to continue to be true that:

$$R_T = R_H + R_O$$

$$\text{where } \; R_H = \sum r_I = r_H N.$$

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[8] A generally comprehensible overview on population development can be found, for example, on Wikimedia Foundation Inc. (Ed.) (2020): https://en.wikipedia.org/wiki/Population_growth, accessed 12 August 2023, and the literature cited therein.

Given p percent increase in the global population and unchanged average per capita consumption, then ceteris paribus, i.e. unchanged consumption of natural resources not caused by humans R_O the entire consumption of natural resources caused by mankind R_H would likewise exhibit proportionate by a factor of $\left(1+\frac{p}{100}\right)$:

$$R_H\left(1+\frac{p}{100}\right)=r_H N\left(1+\frac{p}{100}\right).$$

If human consumption of natural resources is meanwhile to be kept constant even with an increasing global population, then the formal relationship of the *Peren Theorem*[9]

$$R_H \overset{!}{=} r_H N\left(1+\frac{p}{100}\right)$$

determines the following average per capita consumption of natural resources r_H

$$r_H=\frac{R_H}{N}\left(1+\frac{p}{100}\right)^{-1}$$

where human consumption of natural resources as a whole R_H would remain unchanged in relation to the original state prior to the respectively considered period of increase in the global population.

Concomitant with positive population growth of p percent during a certain period, the average per capita consumption of natural resources r_H would have to be proportionately reduced by the factor

$$\left(1+\frac{p}{100}\right)^{-1}$$

In particular, the inhabitants of wealthy national economies, above all the industrialized countries, whose individual human consumption of natural resources r_I is clearly above the average per capita consump-

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[9] The aim is that human consumption of natural resources altogether R_H remains unchanged despite world population growth. Therefore R_H is to be equated with $r_H N(1+\frac{p}{100})$, whereby average human per capita consumption of natural resources r_H is ultimately reduced by the growth factor of the world population $(1+\frac{p}{100})^{-1}$ within the period under consideration.

tion worldwide r_H could by no means continue to maintain their prosperity and lifestyle.

If the global population grows meanwhile with unchanged or even increasing (average) prosperity, as understood and lived today, then the consumption of natural resources would additionally accelerate through an (exponentially) increasing global population with simultaneous shortening of the period of total exhaustion of the natural resources of the Earth.

Appendix A

Financial Mathematical Factors

F. W. Peren, *Math for Business and Economics*,
https://doi.org/10.1007/978-3-662-68860-1

Accumulation Factors $q^n = (1+i)^n$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
1	1.0300	1.0375	1.0400	1.0425	1.0500	1.0600
2	1.0609	1.0764	1.0816	1.0868	1.1025	1.1236
3	1.0927	1.1168	1.1249	1.1330	1.1576	1.1910
4	1.1255	1.1587	1.1699	1.1811	1.2155	1.2625
5	1.1593	1.2021	1.2167	1.2313	1.2763	1.3382
6	1.1941	1.2472	1.2653	1.2837	1.3401	1.4185
7	1.2299	1.2939	1.3159	1.3382	1.4071	1.5036
8	1.2668	1.3425	1.3686	1.3951	1.4775	1.5938
9	1.3048	1.3928	1.4233	1.4544	1.5513	1.6895
10	1.3439	1.4450	1.4802	1.5162	1.6289	1.7908
11	1.3842	1.4992	1.5395	1.5807	1.7103	1.8983
12	1.4258	1.5555	1.6010	1.6478	1.7959	2.0122
13	1.4685	1.6138	1.6651	1.7179	1.8856	2.1329
14	1.5126	1.6743	1.7317	1.7909	1.9799	2.2609
15	1.5580	1.7371	1.8009	1.8670	2.0789	2.3966
16	1.6047	1.8022	1.8730	1.9463	2.1829	2.5404
17	1.6528	1.8698	1.9479	2.0291	2.2920	2.6928
18	1.7024	1.9399	2.0258	2.1153	2.4066	2.8543
19	1.7535	2.0127	2.1068	2.2052	2.527	3.0256
20	1.8061	2.0882	2.1911	2.2989	2.6533	3.2071
21	1.8603	2.1665	2.2788	2.3966	2.7860	3.3996
22	1.9161	2.2477	2.3699	2.4985	2.9253	3.6035
23	1.9736	2.3320	2.4647	2.6047	3.0715	3.8197
24	2.0328	2.4194	2.5633	2.7153	3.2251	4.0489
25	2.0938	2.5102	2.6658	2.8308	3.3864	4.2919
26	2.1566	2.6043	2.7725	2.9511	3.5557	4.5494
27	2.2213	2.7020	2.8834	3.0765	3.7335	4.8223
28	2.2879	2.8033	2.9987	3.2072	3.9201	5.1117
29	2.3566	2.9084	3.1187	3.3435	4.1161	5.4184
30	2.4273	3.0175	3.2434	3.4856	4.3219	5.7435
31	2.5001	3.1306	3.3731	3.6338	4.5380	6.0881
32	2.5751	3.2480	3.5081	3.7882	4.7649	6.4534
33	2.6523	3.3698	3.6484	3.9492	5.0032	6.8406
34	2.7319	3.4962	3.7943	4.1171	5.2533	7.2510
35	2.8139	3.6273	3.9461	4.2920	5.5160	7.6861
36	2.8983	3.7633	4.1039	4.4744	5.7918	8.1473
37	2.9852	3.9045	4.2681	4.6646	6.0814	8.6361
38	3.0748	4.0509	4.4388	4.8628	6.3855	9.1543
39	3.1670	4.2028	4.6164	5.0695	6.7048	9.7035
40	3.2620	4.3604	4.8010	5.2850	7.0400	10.2857

Accumulation Factors $\mathbf{q^n = (1+i)^n}$

n	**i**					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
1	1.0700	1.0800	1.0900	1.1000	1.1200	1.1250
2	1.1449	1.1664	1.1881	1.2100	1.2544	1.2656
3	1.2250	1.2597	1.2950	1.3310	1.4049	1.4238
4	1.3108	1.3605	1.4116	1.4641	1.5735	1.6018
5	1.4026	1.4693	1.5386	1.6105	1.7623	1.8020
6	1.5007	1.5869	1.6771	1.7716	1.9738	2.0273
7	1.6058	1.7138	1.8280	1.9487	2.2107	2.2807
8	1.7182	1.8509	1.9926	2.1436	2.4760	2.5658
9	1.8385	1.9990	2.1719	2.3579	2.7731	2.8865
10	1.9672	2.1589	2.3674	2.5937	3.1058	3.2473
11	2.1049	2.3316	2.5804	2.8531	3.4785	3.6532
12	2.2522	2.5182	2.8127	3.1384	3.8960	4.1099
13	2.4098	2.7196	3.0658	3.4523	4.3635	4.6236
14	2.5785	2.9372	3.3417	3.7975	4.8871	5.2016
15	2.7590	3.1722	3.6425	4.1772	5.4736	5.8518
16	2.9522	3.4259	3.9703	4.5950	6.1304	6.5833
17	3.1588	3.7000	4.3276	5.0545	6.8660	7.4062
18	3.3799	3.9960	4.7171	5.5599	7.6900	8.3319
19	3.6165	4.3157	5.1417	6.1159	8.6128	9.3734
20	3.8697	4.6610	5.6044	6.7275	9.6463	10.5451
21	4.1406	5.0338	6.1088	7.4002	10.8038	11.8632
22	4.4304	5.4365	6.6586	8.1403	12.1003	13.3461
23	4.7405	5.8715	7.2579	8.9543	13.5523	15.0144
24	5.0724	6.3412	7.9111	9.8497	15.1786	16.8912
25	5.4274	6.8485	8.6231	10.8347	17.0001	19.0026
26	5.8074	7.3964	9.3992	11.9182	19.0401	21.3779
27	6.2139	7.9881	10.2451	13.1100	21.3249	24.0502
28	6.6488	8.6271	11.1671	14.4210	23.8839	27.0564
29	7.1143	9.3173	12.1722	15.8631	26.7499	30.4385
30	7.6123	10.0627	13.2677	17.4494	29.9599	34.2433
31	8.1451	10.8677	14.4618	19.1943	33.5551	38.5237
32	8.7153	11.7371	15.7633	21.1138	37.5817	43.3392
33	9.3253	12.6760	17.1820	23.2252	42.0915	48.7566
34	9.9781	13.6901	18.7284	25.5477	47.1425	54.8512
35	10.6766	14.7853	20.4140	28.1024	52.7996	61.7075
36	11.4239	15.9682	22.2512	30.9127	59.1356	69.4210
37	12.2236	17.2456	24.2538	34.0039	66.2318	78.0986
38	13.0793	18.6253	26.4367	37.4043	74.1797	87.8609
39	13.9948	20.1153	28.8160	41.1448	83.0812	98.8436
40	14.9745	21.7245	31.4094	45.2593	93.0510	111.1990

Accumulation Factors $q^n = (1+i)^n$

n	**i**					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
41	3.3599	4.5239	4.9931	5.5096	7.3920	10.9029
42	3.4607	4.6935	5.1928	5.7437	7.7616	11.5570
43	3.5645	4.8695	5.4005	5.9878	8.1497	12.2505
44	3.6715	5.0522	5.6165	6.2423	8.5572	12.9855
45	3.7816	5.2416	5.8412	6.5076	8.9850	13.7646
46	3.8950	5.4382	6.0748	6.7842	9.4343	14.5905
47	4.0119	5.6421	6.3178	7.0725	9.9060	15.4659
48	4.1323	5.8537	6.5705	7.3731	10.4013	16.3939
49	4.2562	6.0732	6.8333	7.6865	10.9213	17.3775
50	4.3839	6.3009	7.1067	8.0131	11.4674	18.4202
51	4.5154	6.5372	7.3910	8.3537	12.0408	19.5254
52	4.6509	6.7824	7.6866	8.7087	12.6428	20.6969
53	4.7904	7.0367	7.9941	9.0789	13.2749	21.9387
54	4.9341	7.3006	8.3138	9.4647	13.9387	23.2550
55	5.0821	7.5744	8.6464	9.8670	14.6356	24.6503
56	5.2346	7.8584	8.9922	10.2863	15.3674	26.1293
57	5.3917	8.1531	9.3519	10.7235	16.1358	27.6971
58	5.5534	8.4588	9.7260	11.1792	16.9426	29.3589
59	5.7200	8.7760	10.1150	11.6543	17.7897	31.1205
60	5.8916	9.1051	10.5196	12.1497	18.6792	32.9877
61	6.0684	9.4466	10.9404	12.6660	19.6131	34.9670
62	6.2504	9.8008	11.3780	13.2043	20.5938	37.0650
63	6.4379	10.1684	11.8332	13.7655	21.6235	39.2889
64	6.6311	10.5497	12.3065	14.3505	22.7047	41.6462
65	6.8300	10.9453	12.7987	14.9604	23.8399	44.1450
66	7.0349	11.3557	13.3107	15.5963	25.0319	46.7937
67	7.2459	11.7816	13.8431	16.2591	26.2835	49.6013
68	7.4633	12.2234	14.3968	16.9501	27.5977	52.5774
69	7.6872	12.6818	14.9727	17.6705	28.9775	55.7320
70	7.9178	13.1573	15.5716	18.4215	30.4264	59.0759
71	8.1554	13.6507	16.1945	19.2044	31.9477	62.6205
72	8.4000	14.1626	16.8423	20.0206	33.5451	66.3777
73	8.6520	14.6937	17.5160	20.8715	35.2224	70.3604
74	8.9116	15.2447	18.2166	21.7585	36.9835	74.5820
75	9.1789	15.8164	18.9453	22.6832	38.8327	79.0569
76	9.4543	16.4095	19.7031	23.6473	40.7743	83.8003
77	9.7379	17.0249	20.4912	24.6523	42.8130	88.8284
78	10.0301	17.6633	21.3108	25.7000	44.9537	94.1581
79	10.3310	18.3257	22.1633	26.7922	47.2014	99.8075
80	10.6409	19.0129	23.0498	27.9309	49.5614	105.7960

Accumulation Factors $\mathbf{q^n = (1+i)^n}$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
41	16.0227	23.4625	34.2363	49.7852	104.2171	125.0989
42	17.1443	25.3395	37.3175	54.7637	116.7231	140.7362
43	18.3444	27.3666	40.6761	60.2401	130.7299	158.3283
44	19.6285	29.5560	44.3370	66.2641	146.4175	178.1193
45	21.0025	31.9204	48.3273	72.8905	163.9876	200.3842
46	22.4726	34.4741	52.6767	80.1795	183.6661	225.4322
47	24.0457	37.2320	57.4176	88.1975	205.7061	253.6113
48	25.7289	40.2106	62.5852	97.0172	230.3908	285.3127
49	27.5299	43.4274	68.2179	106.7190	258.0377	320.9768
50	29.4570	46.9016	74.3575	117.3909	289.0022	361.0989
51	31.5190	50.6537	81.0497	129.1299	323.6825	406.2362
52	33.7253	54.7060	88.3442	142.0429	362.5243	457.0157
53	36.0861	59.0825	96.2951	156.2472	406.0273	514.1427
54	38.6122	63.8091	104.9617	171.8719	454.7505	578.4106
55	41.3150	68.9139	114.4083	189.0591	509.3206	650.7119
56	44.2071	74.4270	124.7050	207.9651	570.4391	732.0509
57	47.3015	80.3811	135.9285	228.7616	638.8918	823.5572
58	50.6127	86.8116	148.1620	251.6377	715.5588	926.5019
59	54.1555	93.7565	161.4966	276.8015	801.4258	1042.3146
60	57.9464	101.2571	176.0313	304.4816	897.5969	1172.6039
61	62.0027	109.3576	191.8741	334.9298	1005.3086	1319.1794
62	66.3429	118.1062	209.1428	368.4228	1125.9456	1484.0769
63	70.9869	127.5547	227.9656	405.2651	1261.0591	1669.5865
64	75.9559	137.7591	248.4825	445.7916	1412.3862	1878.2848
65	81.2729	148.7798	270.8460	490.3707	1581.8725	2113.0704
66	86.9620	160.6822	295.2221	539.4078	1771.6972	2377.2042
67	93.0493	173.5368	321.7921	593.3486	1984.3009	2674.3547
68	99.5627	187.4198	350.7534	652.6834	2222.4170	3008.6490
69	106.5321	202.4133	382.3212	717.9518	2489.1070	3384.7301
70	113.9894	218.6064	416.7301	789.7470	2787.7998	3807.8214
71	121.9686	236.0949	454.2358	868.7217	3122.3358	4283.7991
72	130.5065	254.9825	495.1170	955.5938	3497.0161	4819.2740
73	139.6419	275.3811	539.6775	1051.1532	3916.6580	5421.6832
74	149.4168	297.4116	588.2485	1156.2685	4386.6570	6099.3936
75	159.8760	321.2045	641.1909	1271.8954	4913.0558	6861.8178
76	171.0673	346.9009	698.8981	1399.0849	5502.6225	7719.5450
77	183.0421	374.6530	761.7989	1538.9934	6162.9372	8684.4882
78	195.8550	404.6252	830.3608	1692.8927	6902.4897	9770.0492
79	209.5648	436.9952	905.0933	1862.1820	7730.7885	10991.3054
80	224.2344	471.9548	986.5517	2048.4002	8658.4831	12365.2185

Accumulation Factors $\mathbf{q^n = (1+i)^n}$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
81	10.9601	19.7259	23.9718	29.1180	52.0395	112.1438
82	11.2889	20.4656	24.9307	30.3555	54.6415	118.8724
83	11.6276	21.2331	25.9279	31.6456	57.3736	126.0047
84	11.9764	22.0293	26.9650	32.9905	60.2422	133.5650
85	12.3357	22.8554	28.0436	34.3926	63.2544	141.5789
90	14.3005	27.4745	34.1193	42.3493	80.7304	189.4645
95	16.5782	33.0271	41.5114	52.1466	103.0350	253.5463
100	19.2186	39.7018	50.5049	64.2105	131.5010	339.3021
105	22.2797	47.7260	61.4470	79.0650	167.8300	454.0630
110	25.8282	57.3710	74.7600	97.3570	214.2000	607.6380

Accumulation Factors $\mathbf{q^n = (1+i)^n}$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
81	239.9308	509.7112	1075.3413	2253.2402	9697.5011	13910.8708
82	256.7260	550.4881	1172.1220	2478.5643	10861.2012	15649.7297
83	274.6968	594.5272	1277.6130	2726.4207	12164.5453	17605.9459
84	293.9255	642.0893	1392.5982	2999.0628	13624.2908	19806.6891
85	314.5003	693.4565	1517.9320	3298.9690	15259.2057	22282.5253
90	441.1030	1018.9151	2335.5266	5313.0226	26891.9342	40153.8341
95	618.6697	1497.1205	3593.4971	8556.6760	47392.7766	72358.5129
100	867.7163	2199.7613	5529.0408	13780.6123	83522.2657	130392.3900
105	1217.02	3232.17	8507.11	22193.8	147194.8	234971.3
110	1706.93	4749.12	13089.25	35743.4	259407.5	423425.9

Discount Factors $\mathbf{q^{-n} = (1+i)^{-n}}$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
1	0.97087	0.96386	0.96154	0.95923	0.95238	0.94340
2	0.94260	0.92902	0.92456	0.92013	0.90703	0.89000
3	0.91514	0.89544	0.88900	0.88262	0.86384	0.83962
4	0.88849	0.86307	0.85480	0.84663	0.82270	0.79209
5	0.86261	0.83188	0.82193	0.81212	0.78353	0.74726
6	0.83748	0.80181	0.79031	0.77901	0.74622	0.70496
7	0.81309	0.77283	0.75992	0.74725	0.71068	0.66506
8	0.78941	0.74490	0.73069	0.71679	0.67684	0.62741
9	0.76642	0.71797	0.70259	0.68757	0.64461	0.59190
10	0.74409	0.69202	0.67556	0.65954	0.61391	0.55839
11	0.72242	0.66701	0.64958	0.63265	0.58468	0.52679
12	0.70138	0.64290	0.62460	0.60686	0.55684	0.49697
13	0.68095	0.61966	0.60057	0.58212	0.53032	0.46884
14	0.66112	0.59726	0.57748	0.55839	0.50507	0.44230
15	0.64186	0.57568	0.55526	0.53562	0.48102	0.41727
16	0.62317	0.55487	0.53391	0.51379	0.45811	0.39365
17	0.60502	0.53481	0.51337	0.49284	0.43630	0.37136
18	0.58739	0.51548	0.49363	0.47275	0.41552	0.35034
19	0.57029	0.49685	0.47464	0.45348	0.39573	0.33051
20	0.55368	0.47889	0.45639	0.43499	0.37689	0.31180
21	0.53755	0.46158	0.43883	0.41726	0.35894	0.29416
22	0.52189	0.44490	0.42196	0.40025	0.34185	0.27751
23	0.50669	0.42882	0.40573	0.38393	0.32557	0.26180
24	0.49193	0.41332	0.39012	0.36828	0.31007	0.24698
25	0.47761	0.39838	0.37512	0.35326	0.29530	0.23300
26	0.46369	0.38398	0.36069	0.33886	0.28124	0.21981
27	0.45019	0.37010	0.34682	0.32505	0.26785	0.20737
28	0.43708	0.35672	0.33348	0.31180	0.25509	0.19563
29	0.42435	0.34383	0.32065	0.29908	0.24295	0.18456
30	0.41199	0.33140	0.30832	0.28689	0.23138	0.17411
31	0.39999	0.31942	0.29646	0.27520	0.22036	0.16425
32	0.38834	0.30788	0.28506	0.26398	0.20987	0.15496
33	0.37703	0.29675	0.27409	0.25322	0.19987	0.14619
34	0.36604	0.28603	0.26355	0.24289	0.19035	0.13791
35	0.35538	0.27569	0.25342	0.23299	0.18129	0.13011
36	0.34503	0.26572	0.24367	0.22349	0.17266	0.12274
37	0.33498	0.25612	0.23430	0.21438	0.16444	0.11579
38	0.32523	0.24686	0.22529	0.20564	0.15661	0.10924
39	0.31575	0.23794	0.21662	0.19726	0.14915	0.10306
40	0.30656	0.22934	0.20829	0.18922	0.14205	0.09722

Discount Factors $\mathbf{q^{-n} = (1+i)^{-n}}$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
1	0.93458	0.92593	0.91743	0.90909	0.89286	0.88889
2	0.87344	0.85734	0.84168	0.82645	0.79719	0.79012
3	0.81630	0.79383	0.77218	0.75131	0.71178	0.70233
4	0.76290	0.73503	0.70843	0.68301	0.63552	0.62430
5	0.71299	0.68058	0.64993	0.62092	0.56743	0.55493
6	0.66634	0.63017	0.59627	0.56447	0.50663	0.49327
7	0.62275	0.58349	0.54703	0.51316	0.45235	0.43846
8	0.58201	0.54027	0.50187	0.46651	0.40388	0.38974
9	0.54393	0.50025	0.46043	0.42410	0.36061	0.34644
10	0.50835	0.46319	0.42241	0.38554	0.32197	0.30795
11	0.47509	0.42888	0.38753	0.35049	0.28748	0.27373
12	0.44401	0.39711	0.35553	0.31863	0.25668	0.24332
13	0.41496	0.36770	0.32618	0.28966	0.22917	0.21628
14	0.38782	0.34046	0.29925	0.26333	0.20462	0.19225
15	0.36245	0.31524	0.27454	0.23939	0.18270	0.17089
16	0.33873	0.29189	0.25187	0.21763	0.16312	0.15190
17	0.31657	0.27027	0.23107	0.19784	0.14564	0.13502
18	0.29586	0.25025	0.21199	0.17986	0.13004	0.12002
19	0.27651	0.23171	0.19449	0.16351	0.11611	0.10668
20	0.25842	0.21455	0.17843	0.14864	0.10367	0.09483
21	0.24151	0.19866	0.16370	0.13513	0.09256	0.08429
22	0.22571	0.18394	0.15018	0.12285	0.08264	0.07493
23	0.21095	0.17032	0.13778	0.11168	0.07379	0.06660
24	0.19715	0.15770	0.12640	0.10153	0.06588	0.05920
25	0.18425	0.14602	0.11597	0.09230	0.05882	0.05262
26	0.17220	0.13520	0.10639	0.08391	0.05252	0.04678
27	0.16093	0.12519	0.09761	0.07628	0.04689	0.04158
28	0.15040	0.11591	0.08955	0.06934	0.04187	0.03696
29	0.14056	0.10733	0.08215	0.06304	0.03738	0.03285
30	0.13137	0.09938	0.07537	0.05731	0.03338	0.02920
31	0.12277	0.09202	0.06915	0.05210	0.02980	0.02596
32	0.11474	0.08520	0.06344	0.04736	0.02661	0.02307
33	0.10723	0.07889	0.05820	0.04306	0.02376	0.02051
34	0.10022	0.07305	0.05339	0.03914	0.02121	0.01823
35	0.09366	0.06763	0.04899	0.03558	0.01894	0.01621
36	0.08754	0.06262	0.04494	0.03235	0.01691	0.01440
37	0.08181	0.05799	0.04123	0.02941	0.01510	0.01280
38	0.07646	0.05369	0.03783	0.02673	0.01348	0.01138
39	0.07146	0.04971	0.03470	0.02430	0.01204	0.01012
40	0.06678	0.04603	0.03184	0.02209	0.01075	0.00899

Discount Factors $\mathbf{q^{-n} = (1+i)^{-n}}$

n	**i**					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
41	0.29763	0.22105	0.20028	0.18150	0.13528	0.09172
42	0.28896	0.21306	0.19257	0.17410	0.12884	0.08653
43	0.28054	0.20536	0.18517	0.16700	0.12270	0.08163
44	0.27237	0.19794	0.17805	0.16020	0.11686	0.07701
45	0.26444	0.19078	0.17120	0.15367	0.11130	0.07265
46	0.25674	0.18389	0.16461	0.14740	0.10600	0.06854
47	0.24926	0.17724	0.15828	0.14139	0.10095	0.06466
48	0.24200	0.17083	0.15219	0.13563	0.09614	0.06100
49	0.23495	0.16466	0.14634	0.13010	0.09156	0.05755
50	0.22811	0.15871	0.14071	0.12479	0.08720	0.05429
51	0.22146	0.15297	0.13530	0.11971	0.08305	0.05122
52	0.21501	0.14744	0.13010	0.11483	0.07910	0.04832
53	0.20875	0.14211	0.12509	0.11015	0.07533	0.04558
54	0.20267	0.13698	0.12028	0.10566	0.07174	0.04300
55	0.19677	0.13202	0.11566	0.10135	0.06833	0.04057
56	0.19104	0.12725	0.11121	0.09722	0.06507	0.03827
57	0.18547	0.12265	0.10693	0.09325	0.06197	0.03610
58	0.18007	0.11822	0.10282	0.08945	0.05902	0.03406
59	0.17483	0.11395	0.09886	0.08580	0.05621	0.03213
60	0.16973	0.10983	0.09506	0.08231	0.05354	0.03031
61	0.16479	0.10586	0.09140	0.07895	0.05099	0.02860
62	0.15999	0.10203	0.08789	0.07573	0.04856	0.02698
63	0.15533	0.09834	0.08451	0.07265	0.04625	0.02545
64	0.15081	0.09479	0.08126	0.06968	0.04404	0.02401
65	0.14641	0.09136	0.07813	0.06684	0.04195	0.02265
66	0.14215	0.08806	0.07513	0.06412	0.03995	0.02137
67	0.13801	0.08488	0.07224	0.06150	0.03805	0.02016
68	0.13399	0.08181	0.06946	0.05900	0.03623	0.01902
69	0.13009	0.07885	0.06679	0.05659	0.03451	0.01794
70	0.12630	0.07600	0.06422	0.05428	0.03287	0.01693
71	0.12262	0.07326	0.06175	0.05207	0.03130	0.01597
72	0.11905	0.07061	0.05937	0.04995	0.02981	0.01507
73	0.11558	0.06806	0.05709	0.04791	0.02839	0.01421
74	0.11221	0.06560	0.05490	0.04596	0.02704	0.01341
75	0.10895	0.06323	0.05278	0.04409	0.02575	0.01265
76	0.10577	0.06094	0.05075	0.04229	0.02453	0.01193
77	0.10269	0.05874	0.04880	0.04056	0.02336	0.01126
78	0.09970	0.05661	0.04692	0.03891	0.02225	0.01062
79	0.09680	0.05457	0.04512	0.03732	0.02119	0.01002
80	0.09398	0.05260	0.04338	0.03580	0.02018	0.00945

Discount Factors $\mathbf{q^{-n} = (1+i)^{-n}}$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
41	0.06241	0.04262	0.02921	0.02009	0.00960	0.00799
42	0.05833	0.03946	0.02680	0.01826	0.00857	0.00711
43	0.05451	0.03654	0.02458	0.01660	0.00765	0.00632
44	0.05095	0.03383	0.02255	0.01509	0.00683	0.00561
45	0.04761	0.03133	0.02069	0.01372	0.00610	0.00499
46	0.04450	0.02901	0.01898	0.01247	0.00544	0.00444
47	0.04159	0.02686	0.01742	0.01134	0.00486	0.00394
48	0.03887	0.02487	0.01598	0.01031	0.00434	0.00350
49	0.03632	0.02303	0.01466	0.00937	0.00388	0.00312
50	0.03395	0.02132	0.01345	0.00852	0.00346	0.00277
51	0.03173	0.01974	0.01234	0.00774	0.00309	0.00246
52	0.02965	0.01828	0.01132	0.00704	0.00276	0.00219
53	0.02771	0.01693	0.01038	0.00640	0.00246	0.00194
54	0.02590	0.01567	0.00953	0.00582	0.00220	0.00173
55	0.02420	0.01451	0.00874	0.00529	0.00196	0.00154
56	0.02262	0.01344	0.00802	0.00481	0.00175	0.00137
57	0.02114	0.01244	0.00736	0.00437	0.00157	0.00121
58	0.01976	0.01152	0.00675	0.00397	0.00140	0.00108
59	0.01847	0.01067	0.00619	0.00361	0.00125	0.00096
60	0.01726	0.00988	0.00568	0.00328	0.00111	0.00085
61	0.01613	0.00914	0.00521	0.00299	0.00099	0.00076
62	0.01507	0.00847	0.00478	0.00271	0.00089	0.00067
63	0.01409	0.00784	0.00439	0.00247	0.00079	0.00060
64	0.01317	0.00726	0.00402	0.00224	0.00071	0.00053
65	0.01230	0.00672	0.00369	0.00204	0.00063	0.00047
66	0.01150	0.00622	0.00339	0.00185	0.00056	0.00042
67	0.01075	0.00576	0.00311	0.00169	0.00050	0.00037
68	0.01004	0.00534	0.00285	0.00153	0.00045	0.00033
69	0.00939	0.00494	0.00262	0.00139	0.00040	0.00030
70	0.00877	0.00457	0.00240	0.00127	0.00036	0.00026
71	0.00820	0.00424	0.00220	0.00115	0.00032	0.00023
72	0.00766	0.00392	0.00202	0.00105	0.00029	0.00021
73	0.00716	0.00363	0.00185	0.00095	0.00026	0.00018
74	0.00669	0.00336	0.00170	0.00086	0.00023	0.00016
75	0.00625	0.00311	0.00156	0.00079	0.00020	0.00015
76	0.00585	0.00288	0.00143	0.00071	0.00018	0.00013
77	0.00546	0.00267	0.00131	0.00065	0.00016	0.00012
78	0.00511	0.00247	0.00120	0.00059	0.00014	0.00010
79	0.00477	0.00229	0.00110	0.00054	0.00013	0.00009
80	0.00446	0.00212	0.00101	0.00049	0.00012	0.00008

Discount Factors $q^{-n} = (1+i)^{-n}$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
81	0.09124	0.05069	0.04172	0.03434	0.01922	0.00892
82	0.08858	0.04886	0.04011	0.03294	0.01830	0.00841
83	0.08600	0.04710	0.03857	0.03160	0.01743	0.00794
84	0.08350	0.04539	0.03709	0.03031	0.01660	0.00749
85	0.08107	0.04375	0.03566	0.02908	0.01581	0.00706
90	0.06993	0.03640	0.02931	0.02361	0.01239	0.00528
95	0.06032	0.03028	0.02409	0.01918	0.00971	0.00394
100	0.05203	0.02519	0.01980	0.01557	0.00760	0.00295
105	0.04488	0.02095	0.01627	0.01265	0.00596	0.00220
110	0.03872	0.01743	0.01338	0.01027	0.00467	0.00165

Discount Factors $q^{-n} = (1+i)^{-n}$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
81	0.00417	0.00196	0.00093	0.00044	0.00010	0.00007
82	0.00390	0.00182	0.00085	0.00040	0.00009	0.00006
83	0.00364	0.00168	0.00078	0.00037	0.00008	0.00006
84	0.00340	0.00156	0.00072	0.00033	0.00007	0.00005
85	0.00318	0.00144	0.00066	0.00030	0.00007	0.00004
90	0.00227	0.00098	0.00043	0.00019	0.00004	0.00002
95	0.00162	0.00067	0.00028	0.00012	0.00002	0.00001
100	0.00115	0.00045	0.00018	0.00007	0.00001	0.00001
105	0.00082	0.00031	0.00012	0.00005	0.00001	0.00000
110	0.00059	0.00021	0.00008	0.00003	0.00000	0.00000

Repayment Factor $\frac{\mathbf{q-1}}{\mathbf{q^n-1}} = \frac{\mathbf{i}}{\mathbf{(1+i)^n-1}}$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.49261	0.49080	0.49020	0.48960	0.48780	0.48544
3	0.32353	0.32114	0.32035	0.31956	0.31721	0.31411
4	0.23903	0.23637	0.23549	0.23462	0.23201	0.22859
5	0.18835	0.18555	0.18463	0.18371	0.18097	0.17740
6	0.15460	0.15171	0.15076	0.14982	0.14702	0.14336
7	0.13051	0.12757	0.12661	0.12565	0.12282	0.11914
8	0.11246	0.10950	0.10853	0.10756	0.10472	0.10104
9	0.09843	0.09547	0.09449	0.09353	0.09069	0.08702
10	0.08723	0.08426	0.08329	0.08233	0.07950	0.07587
11	0.07808	0.07512	0.07415	0.07319	0.07039	0.06679
12	0.07046	0.06751	0.06655	0.06560	0.06283	0.05928
13	0.06403	0.06110	0.06014	0.05920	0.05646	0.05296
14	0.05853	0.05561	0.05467	0.05374	0.05102	0.04758
15	0.05377	0.05088	0.04994	0.04902	0.04634	0.04296
16	0.04961	0.04674	0.04582	0.04491	0.04227	0.03895
17	0.04595	0.04311	0.04220	0.04130	0.03870	0.03544
18	0.04271	0.03990	0.03899	0.03811	0.03555	0.03236
19	0.03981	0.03703	0.03614	0.03526	0.03275	0.02962
20	0.03722	0.03446	0.03358	0.03272	0.03024	0.02718
21	0.03487	0.03215	0.03128	0.03043	0.02800	0.02500
22	0.03275	0.03006	0.02920	0.02836	0.02597	0.02305
23	0.03081	0.02815	0.02731	0.02649	0.02414	0.02128
24	0.02905	0.02642	0.02559	0.02478	0.02247	0.01968
25	0.02743	0.02483	0.02401	0.02321	0.02095	0.01823
26	0.02594	0.02337	0.02257	0.02178	0.01956	0.01690
27	0.02456	0.02203	0.02124	0.02047	0.01829	0.01570
28	0.02329	0.02080	0.02001	0.01925	0.01712	0.01459
29	0.02211	0.01965	0.01888	0.01813	0.01605	0.01358
30	0.02102	0.01859	0.01783	0.01710	0.01505	0.01265
31	0.02000	0.01760	0.01686	0.01614	0.01413	0.01179
32	0.01905	0.01668	0.01595	0.01524	0.01328	0.01100
33	0.01816	0.01582	0.01510	0.01441	0.01249	0.01027
34	0.01732	0.01502	0.01431	0.01363	0.01176	0.00960
35	0.01654	0.01427	0.01358	0.01291	0.01107	0.00897
36	0.01580	0.01357	0.01289	0.01223	0.01043	0.00839
37	0.01511	0.01291	0.01224	0.01160	0.00984	0.00786
38	0.01446	0.01229	0.01163	0.01100	0.00928	0.00736
39	0.01384	0.01171	0.01106	0.01044	0.00876	0.00689
40	0.01326	0.01116	0.01052	0.00992	0.00828	0.00646

Repayment Factor $\frac{q-1}{q^n-1} = \frac{i}{(1+i)^n-1}$

n	**i**					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.48309	0.48077	0.47847	0.47619	0.47170	0.47059
3	0.31105	0.30803	0.30505	0.30211	0.29635	0.29493
4	0.22523	0.22192	0.21867	0.21547	0.20923	0.20771
5	0.17389	0.17046	0.16709	0.16380	0.15741	0.15585
6	0.13980	0.13632	0.13292	0.12961	0.12323	0.12168
7	0.11555	0.11207	0.10869	0.10541	0.09912	0.09760
8	0.09747	0.09401	0.09067	0.08744	0.08130	0.07983
9	0.08349	0.08008	0.07680	0.07364	0.06768	0.06626
10	0.07238	0.06903	0.06582	0.06275	0.05698	0.05562
11	0.06336	0.06008	0.05695	0.05396	0.04842	0.04711
12	0.05590	0.05270	0.04965	0.04676	0.04144	0.04019
13	0.04965	0.04652	0.04357	0.04078	0.03568	0.03450
14	0.04434	0.04130	0.03843	0.03575	0.03087	0.02975
15	0.03979	0.03683	0.03406	0.03147	0.02682	0.02576
16	0.03586	0.03298	0.03030	0.02782	0.02339	0.02239
17	0.03243	0.02963	0.02705	0.02466	0.02046	0.01951
18	0.02941	0.02670	0.02421	0.02193	0.01794	0.01705
19	0.02675	0.02413	0.02173	0.01955	0.01576	0.01493
20	0.02439	0.02185	0.01955	0.01746	0.01388	0.01310
21	0.02229	0.01983	0.01762	0.01562	0.01224	0.01151
22	0.02041	0.01803	0.01590	0.01401	0.01081	0.01012
23	0.01871	0.01642	0.01438	0.01257	0.00956	0.00892
24	0.01719	0.01498	0.01302	0.01130	0.00846	0.00787
25	0.01581	0.01368	0.01181	0.01017	0.00750	0.00694
26	0.01456	0.01251	0.01072	0.00916	0.00665	0.00613
27	0.01343	0.01145	0.00973	0.00826	0.00590	0.00542
28	0.01239	0.01049	0.00885	0.00745	0.00524	0.00480
29	0.01145	0.00962	0.00806	0.00673	0.00466	0.00425
30	0.01059	0.00883	0.00734	0.00608	0.00414	0.00376
31	0.00980	0.00811	0.00669	0.00550	0.00369	0.00333
32	0.00907	0.00745	0.00610	0.00497	0.00328	0.00295
33	0.00841	0.00685	0.00556	0.00450	0.00292	0.00262
34	0.00780	0.00630	0.00508	0.00407	0.00260	0.00232
35	0.00723	0.00580	0.00464	0.00369	0.00232	0.00206
36	0.00672	0.00534	0.00424	0.00334	0.00206	0.00183
37	0.00624	0.00492	0.00387	0.00303	0.00184	0.00162
38	0.00580	0.00454	0.00354	0.00275	0.00164	0.00144
39	0.00539	0.00419	0.00324	0.00249	0.00146	0.00128
40	0.00501	0.00386	0.00296	0.00226	0.00130	0.00113

Repayment Factor $\frac{q-1}{q^n-1} = \frac{i}{(1+i)^n-1}$

n	**i**					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
41	0.01271	0.01064	0.01002	0.00942	0.00782	0.00606
42	0.01219	0.01015	0.00954	0.00896	0.00739	0.00568
43	0.01170	0.00969	0.00909	0.00852	0.00699	0.00533
44	0.01123	0.00925	0.00866	0.00811	0.00662	0.00501
45	0.01079	0.00884	0.00826	0.00772	0.00626	0.00470
46	0.01036	0.00845	0.00788	0.00735	0.00593	0.00441
47	0.00996	0.00808	0.00752	0.00700	0.00561	0.00415
48	0.00958	0.00773	0.00718	0.00667	0.00532	0.00390
49	0.00921	0.00739	0.00686	0.00636	0.00504	0.00366
50	0.00887	0.00707	0.00655	0.00606	0.00478	0.00344
51	0.00853	0.00677	0.00626	0.00578	0.00453	0.00324
52	0.00822	0.00649	0.00598	0.00551	0.00429	0.00305
53	0.00791	0.00621	0.00572	0.00526	0.00407	0.00287
54	0.00763	0.00595	0.00547	0.00502	0.00386	0.00270
55	0.00735	0.00570	0.00523	0.00479	0.00367	0.00254
56	0.00708	0.00547	0.00500	0.00458	0.00348	0.00239
57	0.00683	0.00524	0.00479	0.00437	0.00330	0.00225
58	0.00659	0.00503	0.00458	0.00418	0.00314	0.00212
59	0.00636	0.00482	0.00439	0.00399	0.00298	0.00199
60	0.00613	0.00463	0.00420	0.00381	0.00283	0.00188
61	0.00592	0.00444	0.00402	0.00364	0.00269	0.00177
62	0.00571	0.00426	0.00385	0.00348	0.00255	0.00166
63	0.00552	0.00409	0.00369	0.00333	0.00242	0.00157
64	0.00533	0.00393	0.00354	0.00318	0.00230	0.00148
65	0.00515	0.00377	0.00339	0.00304	0.00219	0.00139
66	0.00497	0.00362	0.00325	0.00291	0.00208	0.00131
67	0.00480	0.00348	0.00311	0.00279	0.00198	0.00123
68	0.00464	0.00334	0.00299	0.00266	0.00188	0.00116
69	0.00449	0.00321	0.00286	0.00255	0.00179	0.00110
70	0.00434	0.00308	0.00275	0.00244	0.00170	0.00103
71	0.00419	0.00296	0.00263	0.00233	0.00162	0.00097
72	0.00405	0.00285	0.00252	0.00223	0.00154	0.00092
73	0.00392	0.00274	0.00242	0.00214	0.00146	0.00087
74	0.00379	0.00263	0.00232	0.00205	0.00139	0.00082
75	0.00367	0.00253	0.00223	0.00196	0.00132	0.00077
76	0.00355	0.00243	0.00214	0.00188	0.00126	0.00072
77	0.00343	0.00234	0.00205	0.00180	0.00120	0.00068
78	0.00332	0.00225	0.00197	0.00172	0.00114	0.00064
79	0.00322	0.00216	0.00189	0.00165	0.00108	0.00061
80	0.00311	0.00208	0.00181	0.00158	0.00103	0.00057

Repayment Factor $\frac{q-1}{q^n-1} = \frac{i}{(1+i)^n-1}$

n	i					
	0.07	0.08	0.09	0.10	0.12	0.125
41	0.00466	0.00356	0.00271	0.00205	0.00116	0.00101
42	0.00434	0.00329	0.00248	0.00186	0.00104	0.00089
43	0.00404	0.00303	0.00227	0.00169	0.00092	0.00079
44	0.00376	0.00280	0.00208	0.00153	0.00083	0.00071
45	0.00350	0.00259	0.00190	0.00139	0.00074	0.00063
46	0.00326	0.00239	0.00174	0.00126	0.00066	0.00056
47	0.00304	0.00221	0.00160	0.00115	0.00059	0.00049
48	0.00283	0.00204	0.00146	0.00104	0.00052	0.00044
49	0.00264	0.00189	0.00134	0.00095	0.00047	0.00039
50	0.00246	0.00174	0.00123	0.00086	0.00042	0.00035
51	0.00229	0.00161	0.00112	0.00078	0.00037	0.00031
52	0.00214	0.00149	0.00103	0.00071	0.00033	0.00027
53	0.00200	0.00138	0.00094	0.00064	0.00030	0.00024
54	0.00186	0.00127	0.00087	0.00059	0.00026	0.00022
55	0.00174	0.00118	0.00079	0.00053	0.00024	0.00019
56	0.00162	0.00109	0.00073	0.00048	0.00021	0.00017
57	0.00151	0.00101	0.00067	0.00044	0.00019	0.00015
58	0.00141	0.00093	0.00061	0.00040	0.00017	0.00014
59	0.00132	0.00086	0.00056	0.00036	0.00015	0.00012
60	0.00123	0.00080	0.00051	0.00033	0.00013	0.00011
61	0.00115	0.00074	0.00047	0.00030	0.00012	0.00009
62	0.00107	0.00068	0.00043	0.00027	0.00011	0.00008
63	0.00100	0.00063	0.00040	0.00025	0.00010	0.00007
64	0.00093	0.00058	0.00036	0.00022	0.00009	0.00007
65	0.00087	0.00054	0.00033	0.00020	0.00008	0.00006
66	0.00081	0.00050	0.00031	0.00019	0.00007	0.00005
67	0.00076	0.00046	0.00028	0.00017	0.00006	0.00005
68	0.00071	0.00043	0.00026	0.00015	0.00005	0.00004
69	0.00066	0.00040	0.00024	0.00014	0.00005	0.00004
70	0.00062	0.00037	0.00022	0.00013	0.00004	0.00003
71	0.00058	0.00034	0.00020	0.00012	0.00004	0.00003
72	0.00054	0.00031	0.00018	0.00010	0.00003	0.00003
73	0.00050	0.00029	0.00017	0.00010	0.00003	0.00002
74	0.00047	0.00027	0.00015	0.00009	0.00003	0.00002
75	0.00044	0.00025	0.00014	0.00008	0.00002	0.00002
76	0.00041	0.00023	0.00013	0.00007	0.00002	0.00002
77	0.00038	0.00021	0.00012	0.00007	0.00002	0.00001
78	0.00036	0.00020	0.00011	0.00006	0.00002	0.00001
79	0.00034	0.00018	0.00010	0.00005	0.00002	0.00001
80	0.00031	0.00017	0.00009	0.00005	0.00001	0.00001

Repayment Factor $\frac{q-1}{q^n-1} = \frac{i}{(1+i)^n-1}$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
85	0.00265	0.00172	0.00148	0.00127	0.0008	0.00043
90	0.00226	0.00142	0.00121	0.00103	0.00063	0.00032
95	0.00193	0.00117	0.00099	0.00083	0.00049	0.00024
100	0.00165	0.00097	0.00081	0.00067	0.00038	0.00018
105	0.00141	0.00080	0.00066	0.00054	0.00030	0.00013

Repayment Factor $\frac{q-1}{q^n-1} = \frac{i}{(1+i)^n-1}$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
85	0.00022	0.00012	0.00006	0.00003	0.00001	0.00001
90	0.00016	0.00008	0.00004	0.00002	0.00000	0.00000
95	0.00011	0.00005	0.00003	0.00001	0.00000	0.00000
100	0.00008	0.00004	0.00002	0.00001	0.00000	0.00000
105	0.00006	0.00002	0.00001	0.00000	0.00000	0.00000

Annuity Value Factors (in advance) $\frac{q^n-1}{q^n\cdot(q-1)}\cdot q=\frac{(1+i)^n-1}{i\cdot(1+i)^n}\cdot(1+i)$

n	**i**					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.97087	1.96386	1.96154	1.95923	1.95238	1.94340
3	2.91347	2.89287	2.88609	2.87936	2.85941	2.83339
4	3.82861	3.78831	3.77509	3.76198	3.72325	3.67301
5	4.71710	4.65138	4.62990	4.60861	4.54595	4.46511
6	5.57971	5.48326	5.45182	5.42073	5.32948	5.21236
7	6.41719	6.28507	6.24214	6.19974	6.07569	5.91732
8	7.23028	7.05790	7.00205	6.94699	6.78637	6.58238
9	8.01969	7.80280	7.73274	7.66378	7.46321	7.20979
10	8.78611	8.52077	8.43533	8.35135	8.10782	7.80169
11	9.53020	9.21279	9.11090	9.01089	8.72173	8.36009
12	10.25262	9.87979	9.76048	9.64354	9.30641	8.88687
13	10.95400	10.52269	10.38507	10.25039	9.86325	9.38384
14	11.63496	11.14236	10.98565	10.83251	10.39357	9.85268
15	12.29607	11.73962	11.56312	11.39090	10.89864	10.29498
16	12.93794	12.31530	12.11839	11.92652	11.37966	10.71225
17	13.56110	12.87017	12.65230	12.44031	11.83777	11.10590
18	14.16612	13.40498	13.16567	12.93315	12.27407	11.47726
19	14.75351	13.92046	13.65930	13.40590	12.68959	11.82760
20	15.32380	14.41731	14.13394	13.85938	13.08532	12.15812
21	15.87747	14.89620	14.59033	14.29437	13.46221	12.46992
22	16.41502	15.35779	15.02916	14.71162	13.82115	12.76408
23	16.93692	15.80269	15.45112	15.11187	14.16300	13.04158
24	17.44361	16.23151	15.85684	15.49580	14.48857	13.30338
25	17.93554	16.64482	16.24696	15.86407	14.79864	13.55036
26	18.41315	17.04320	16.62208	16.21734	15.09394	13.78336
27	18.87684	17.42718	16.98277	16.55620	15.37519	14.00317
28	19.32703	17.79729	17.32959	16.88124	15.64303	14.21053
29	19.76411	18.15401	17.66306	17.19304	15.89813	14.40616
30	20.18845	18.49784	17.98371	17.49213	16.14107	14.59072
31	20.60044	18.82925	18.29203	17.77902	16.37245	14.76483
32	21.00043	19.14867	18.58849	18.05421	16.59281	14.92909
33	21.38877	19.45655	18.87355	18.31819	16.80268	15.08404
34	21.76579	19.75330	19.14765	18.57141	17.00255	15.23023
35	22.13184	20.03933	19.41120	18.81430	17.19290	15.36814
36	22.48722	20.31501	19.66461	19.04729	17.37419	15.49825
37	22.83225	20.58074	19.90828	19.27078	17.54685	15.62099
38	23.16724	20.83685	20.14258	19.48516	17.71129	15.73678
39	23.49246	21.08371	20.36786	19.69080	17.86789	15.84602
40	23.80822	21.32165	20.58448	19.88806	18.01704	15.94907

Annuity Value Factors (in advance) $\frac{q^n-1}{q^n\cdot(q-1)}\cdot q = \frac{(1+i)^n-1}{i\cdot(1+i)^n}\cdot(1+i)$

n	i					
	0.07	0.08	0.09	0.10	0.12	0.125
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.93458	1.92593	1.91743	1.90909	1.89286	1.88889
3	2.80802	2.78326	2.75911	2.73554	2.69005	2.67901
4	3.62432	3.57710	3.53129	3.48685	3.40183	3.38134
5	4.38721	4.31213	4.23972	4.16987	4.03735	4.00564
6	5.10020	4.99271	4.88965	4.79079	4.60478	4.56057
7	5.76654	5.62288	5.48592	5.35526	5.11141	5.05384
8	6.38929	6.20637	6.03295	5.86842	5.56376	5.49230
9	6.97130	6.74664	6.53482	6.33493	5.96764	5.88205
10	7.51523	7.24689	6.99525	6.75902	6.32825	6.22848
11	8.02358	7.71008	7.41766	7.14457	6.65022	6.53643
12	8.49867	8.13896	7.80519	7.49506	6.93770	6.81016
13	8.94269	8.53608	8.16073	7.81369	7.19437	7.05348
14	9.35765	8.90378	8.48690	8.10336	7.42355	7.26976
15	9.74547	9.24424	8.78615	8.36669	7.62817	7.46201
16	10.10791	9.55948	9.06069	8.60608	7.81086	7.63289
17	10.44665	9.85137	9.31256	8.82371	7.97399	7.78479
18	10.76322	10.12164	9.54363	9.02155	8.11963	7.91982
19	11.05909	10.37189	9.75563	9.20141	8.24967	8.03984
20	11.33560	10.60360	9.95011	9.36492	8.36578	8.14652
21	11.59401	10.81815	10.12855	9.51356	8.46944	8.24135
22	11.83553	11.01680	10.29224	9.64869	8.56200	8.32565
23	12.06124	11.20074	10.44243	9.77154	8.64465	8.40058
24	12.27219	11.37106	10.58021	9.88322	8.71843	8.46718
25	12.46933	11.52876	10.70661	9.98474	8.78432	8.52638
26	12.65358	11.67478	10.82258	10.07704	8.84314	8.57901
27	12.82578	11.80998	10.92897	10.16095	8.89566	8.62578
28	12.98671	11.93516	11.02658	10.23722	8.94255	8.66736
29	13.13711	12.05108	11.11613	10.30657	8.98442	8.70432
30	13.27767	12.15841	11.19828	10.36961	9.02181	8.73717
31	13.40904	12.25778	11.27365	10.42691	9.05518	8.76638
32	13.53181	12.34980	11.34280	10.47901	9.08499	8.79234
33	13.64656	12.43500	11.40624	10.52638	9.11159	8.81541
34	13.75379	12.51389	11.46444	10.56943	9.13535	8.83592
35	13.85401	12.58693	11.51784	10.60857	9.15656	8.85415
36	13.94767	12.65457	11.56682	10.64416	9.17550	8.87036
37	14.03521	12.71719	11.61176	10.67651	9.19241	8.88476
38	14.11702	12.77518	11.65299	10.70592	9.20751	8.89757
39	14.19347	12.82887	11.69082	10.73265	9.22099	8.90895
40	14.26493	12.87858	11.72552	10.75696	9.23303	8.91906

Annuity Value Factors (in advance) $\frac{q^n-1}{q^n\cdot(q-1)}\cdot q=\frac{(1+i)^n-1}{i\cdot(1+i)^n}\cdot(1+i)$

n	i					
	0.03	0.0375	0.04	0.0425	0.05	0.06
41	24.11477	21.55099	20.79277	20.07727	18.15909	16.04630
42	24.41240	21.77204	20.99305	20.25878	18.29437	16.13802
43	24.70136	21.98510	21.18563	20.43288	18.42321	16.22454
44	24.98190	22.19046	21.37079	20.59988	18.54591	16.30617
45	25.25427	22.38839	21.54884	20.76008	18.66277	16.38318
46	25.51871	22.57917	21.72004	20.91375	18.77407	16.45583
47	25.77545	22.76306	21.88465	21.06115	18.88007	16.52437
48	26.02471	22.94030	22.04294	21.20254	18.98102	16.58903
49	26.26671	23.11113	22.19513	21.33817	19.07716	16.65003
50	26.50166	23.27579	22.34147	21.46827	19.16872	16.70757
51	26.72976	23.43449	22.48218	21.59306	19.25593	16.76186
52	26.95123	23.58746	22.61749	21.71277	19.33898	16.81308
53	27.16624	23.73490	22.74758	21.82760	19.41807	16.86139
54	27.37499	23.87702	22.87267	21.93774	19.49340	16.90697
55	27.57766	24.01399	22.99296	22.04340	19.56515	16.94998
56	27.77443	24.14602	23.10861	22.14475	19.63347	16.99054
57	27.96546	24.27327	23.21982	22.24196	19.69854	17.02881
58	28.15094	24.39592	23.32675	22.33522	19.76052	17.06492
59	28.33101	24.51414	23.42957	22.42467	19.81954	17.09898
60	28.50583	24.62809	23.52843	22.51047	19.87575	17.13111
61	28.67556	24.73792	23.62349	22.59278	19.92929	17.16143
62	28.84035	24.84377	23.71489	22.67173	19.98028	17.19003
63	29.00034	24.94581	23.80278	22.74746	20.02883	17.21701
64	29.15567	25.04415	23.88729	22.82011	20.07508	17.24246
65	29.30648	25.13894	23.96855	22.88979	20.11912	17.26647
66	29.45289	25.23030	24.04668	22.95664	20.16107	17.28912
67	29.59504	25.31837	24.12181	23.02075	20.20102	17.31049
68	29.73305	25.40324	24.19405	23.08226	20.23907	17.33065
69	29.86704	25.48505	24.26351	23.14125	20.27530	17.34967
70	29.99712	25.56391	24.33030	23.19785	20.30981	17.36762
71	30.12342	25.63991	24.39451	23.25213	20.34268	17.38454
72	30.24604	25.71317	24.45626	23.30420	20.37398	17.40051
73	30.36509	25.78378	24.51564	23.35415	20.40379	17.41558
74	30.48067	25.85183	24.57273	23.40206	20.43218	17.42979
75	30.59288	25.91743	24.62762	23.44802	20.45922	17.44320
76	30.70183	25.98065	24.68041	23.49211	20.48497	17.45585
77	30.80760	26.04159	24.73116	23.53440	20.50950	17.46778
78	30.91029	26.10033	24.77996	23.57496	20.53285	17.47904
79	31.00999	26.15695	24.82689	23.61387	20.55510	17.48966
80	31.10679	26.21151	24.87201	23.65119	20.57628	17.49968

Annuity Value Factors (in advance) $\frac{q^n-1}{q^n\cdot(q-1)}\cdot q=\frac{(1+i)^n-1}{i\cdot(1+i)^n}\cdot(1+i)$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
41	14.33171	12.92461	11.75736	10.77905	9.24378	8.92806
42	14.39412	12.96723	11.78657	10.79914	9.25337	8.93605
43	14.45245	13.00670	11.81337	10.81740	9.26194	8.94316
44	14.50696	13.04324	11.83795	10.83400	9.26959	8.94947
45	14.55791	13.07707	11.86051	10.84909	9.27642	8.95509
46	14.60552	13.10840	11.88120	10.86281	9.28252	8.96008
47	14.65002	13.13741	11.90018	10.87528	9.28796	8.96451
48	14.69161	13.16427	11.91760	10.88662	9.29282	8.96846
49	14.73047	13.18914	11.93358	10.89693	9.29716	8.97196
50	14.76680	13.21216	11.94823	10.90630	9.30104	8.97508
51	14.80075	13.23348	11.96168	10.91481	9.30450	8.97785
52	14.83247	13.25323	11.97402	10.92256	9.30759	8.98031
53	14.86212	13.27151	11.98534	10.92960	9.31035	8.98250
54	14.88984	13.28843	11.99573	10.93600	9.31281	8.98444
55	14.91573	13.30410	12.00525	10.94182	9.31501	8.98617
56	14.93994	13.31861	12.01399	10.94711	9.31697	8.98771
57	14.96256	13.33205	12.02201	10.95191	9.31872	8.98907
58	14.98370	13.34449	12.02937	10.95629	9.32029	8.99029
59	15.00346	13.35601	12.03612	10.96026	9.32169	8.99137
60	15.02192	13.36668	12.04231	10.96387	9.32294	8.99232
61	15.03918	13.37655	12.04799	10.96716	9.32405	8.99318
62	15.05531	13.38570	12.05320	10.97014	9.32504	8.99394
63	15.07038	13.39416	12.05798	10.97286	9.32593	8.99461
64	15.08447	13.40200	12.06237	10.97532	9.32673	8.99521
65	15.09764	13.40926	12.06640	10.97757	9.32743	8.99574
66	15.10994	13.41598	12.07009	10.97961	9.32807	8.99621
67	15.12144	13.42221	12.07347	10.98146	9.32863	8.99663
68	15.13219	13.42797	12.07658	10.98315	9.32913	8.99701
69	15.14223	13.43330	12.07943	10.98468	9.32958	8.99734
70	15.15162	13.43825	12.08205	10.98607	9.32999	8.99764
71	15.16039	13.44282	12.08445	10.98734	9.33034	8.99790
72	15.16859	13.44706	12.08665	10.98849	9.33066	8.99813
73	15.17625	13.45098	12.08867	10.98954	9.33095	8.99834
74	15.18341	13.45461	12.09052	10.99049	9.33121	8.99852
75	15.19010	13.45797	12.09222	10.99135	9.33143	8.99869
76	15.19636	13.46108	12.09378	10.99214	9.33164	8.99883
77	15.20220	13.46397	12.09521	10.99285	9.33182	8.99896
78	15.20767	13.46664	12.09653	10.99350	9.33198	8.99908
79	15.21277	13.46911	12.09773	10.99409	9.33213	8.99918
80	15.21755	13.47140	12.09883	10.99463	9.33226	8.99927

Annuity Value Factors (in advance) $\frac{q^n-1}{q^n\cdot(q-1)}\cdot q=\frac{(1+i)^n-1}{i\cdot(1+i)^n}\cdot(1+i)$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
85	31.55009	26.45616	25.07287	23.81619	20.66801	17.54188
90	31.93248	26.65967	25.23797	23.95019	20.73987	17.57342
95	32.26234	26.82897	25.37367	24.05902	20.79619	17.59699
100	32.54687	26.96981	25.48520	24.14740	20.84031	17.61460
105	32.79232	27.08696	25.57687	24.21917	20.87488	17.62776

Annuity Value Factors (in advance) $\frac{q^n-1}{q^n\cdot(q-1)}\cdot q=\frac{(1+i)^n-1}{i\cdot(1+i)^n}\cdot(1+i)$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
85	15.23711	13.48053	12.10313	10.99667	9.33272	8.99960
90	15.25106	13.48675	12.10593	10.99793	9.33299	8.99978
95	15.26101	13.49098	12.10774	10.99871	9.33314	8.99988
100	15.26810	13.49386	12.10892	10.99920	9.33322	8.99993
105	15.27315	13.49582	12.10969	10.99950	9.33327	8.99996

Annuity Value Factors (in arrears) $\frac{q^n-1}{q^n\cdot(q-1)} = \frac{(1+i)^n-1}{i\cdot(1+i)^n}$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
1	0.97087	0.96386	0.96154	0.95923	0.95238	0.94340
2	1.91347	1.89287	1.88609	1.87936	1.85941	1.83339
3	2.82861	2.78831	2.77509	2.76198	2.72325	2.67301
4	3.71710	3.65138	3.62990	3.60861	3.54595	3.46511
5	4.57971	4.48326	4.45182	4.42073	4.32948	4.21236
6	5.41719	5.28507	5.24214	5.19974	5.07569	4.91732
7	6.23028	6.05790	6.00205	5.94699	5.78637	5.58238
8	7.01969	6.80280	6.73274	6.66378	6.46321	6.20979
9	7.78611	7.52077	7.43533	7.35135	7.10782	6.80169
10	8.53020	8.21279	8.11090	8.01089	7.72173	7.36009
11	9.25262	8.87979	8.76048	8.64354	8.30641	7.88687
12	9.95400	9.52269	9.38507	9.25039	8.86325	8.38384
13	10.63496	10.14236	9.98565	9.83251	9.39357	8.85268
14	11.29607	10.73962	10.56312	10.39090	9.89864	9.29498
15	11.93794	11.31530	11.11839	10.92652	10.37966	9.71225
16	12.56110	11.87017	11.65230	11.44031	10.83777	10.10590
17	13.16612	12.40498	12.16567	11.93315	11.27407	10.47726
18	13.75351	12.92046	12.65930	12.40590	11.68959	10.82760
19	14.32380	13.41731	13.13394	12.85938	12.08532	11.15812
20	14.87747	13.89620	13.59033	13.29437	12.46221	11.46992
21	15.41502	14.35779	14.02916	13.71162	12.82115	11.76408
22	15.93692	14.80269	14.45112	14.11187	13.16300	12.04158
23	16.44361	15.23151	14.85684	14.49580	13.48857	12.30338
24	16.93554	15.64482	15.24696	14.86407	13.79864	12.55036
25	17.41315	16.04320	15.62208	15.21734	14.09394	12.78336
26	17.87684	16.42718	15.98277	15.55620	14.37519	13.00317
27	18.32703	16.79729	16.32959	15.88124	14.64303	13.21053
28	18.76411	17.15401	16.66306	16.19304	14.89813	13.40616
29	19.18845	17.49784	16.98371	16.49213	15.14107	13.59072
30	19.60044	17.82925	17.29203	16.77902	15.37245	13.76483
31	20.00043	18.14867	17.58849	17.05421	15.59281	13.92909
32	20.38877	18.45655	17.87355	17.31819	15.80268	14.08404
33	20.76579	18.75330	18.14765	17.57141	16.00255	14.23023
34	21.13184	19.03933	18.41120	17.81430	16.19290	14.36814
35	21.48722	19.31501	18.66461	18.04729	16.37419	14.49825
36	21.83225	19.58074	18.90828	18.27078	16.54685	14.62099
37	22.16724	19.83685	19.14258	18.48516	16.71129	14.73678
38	22.49246	20.08371	19.36786	18.69080	16.86789	14.84602
39	22.80822	20.32165	19.58448	18.88806	17.01704	14.94907
40	23.11477	20.55099	19.79277	19.07727	17.15909	15.04630

Annuity Value Factors (in arrears) $\frac{q^n-1}{q^n\cdot(q-1)} = \frac{(1+i)^n-1}{i\cdot(1+i)^n}$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
1	0.93458	0.92593	0.91743	0.90909	0.89286	0.88889
2	1.80802	1.78326	1.75911	1.73554	1.69005	1.67901
3	2.62432	2.57710	2.53129	2.48685	2.40183	2.38134
4	3.38721	3.31213	3.23972	3.16987	3.03735	3.00564
5	4.10020	3.99271	3.88965	3.79079	3.60478	3.56057
6	4.76654	4.62288	4.48592	4.35526	4.11141	4.05384
7	5.38929	5.20637	5.03295	4.86842	4.56376	4.49230
8	5.97130	5.74664	5.53482	5.33493	4.96764	4.88205
9	6.51523	6.24689	5.99525	5.75902	5.32825	5.22848
10	7.02358	6.71008	6.41766	6.14457	5.65022	5.53643
11	7.49867	7.13896	6.80519	6.49506	5.93770	5.81016
12	7.94269	7.53608	7.16073	6.81369	6.19437	6.05348
13	8.35765	7.90378	7.48690	7.10336	6.42355	6.26976
14	8.74547	8.24424	7.78615	7.36669	6.62817	6.46201
15	9.10791	8.55948	8.06069	7.60608	6.81086	6.63289
16	9.44665	8.85137	8.31256	7.82371	6.97399	6.78479
17	9.76322	9.12164	8.54363	8.02155	7.11963	6.91982
18	10.05909	9.37189	8.75563	8.20141	7.24967	7.03984
19	10.33560	9.60360	8.95011	8.36492	7.36578	7.14652
20	10.59401	9.81815	9.12855	8.51356	7.46944	7.24135
21	10.83553	10.01680	9.29224	8.64869	7.56200	7.32565
22	11.06124	10.20074	9.44243	8.77154	7.64465	7.40058
23	11.27219	10.37106	9.58021	8.88322	7.71843	7.46718
24	11.46933	10.52876	9.70661	8.98474	7.78432	7.52638
25	11.65358	10.67478	9.82258	9.07704	7.84314	7.57901
26	11.82578	10.80998	9.92897	9.16095	7.89566	7.62578
27	11.98671	10.93516	10.02658	9.23722	7.94255	7.66736
28	12.13711	11.05108	10.11613	9.30657	7.98442	7.70432
29	12.27767	11.15841	10.19828	9.36961	8.02181	7.73717
30	12.40904	11.25778	10.27365	9.42691	8.05518	7.76638
31	12.53181	11.34980	10.34280	9.47901	8.08499	7.79234
32	12.64656	11.43500	10.40624	9.52638	8.11159	7.81541
33	12.75379	11.51389	10.46444	9.56943	8.13535	7.83592
34	12.85401	11.58693	10.51784	9.60857	8.15656	7.85415
35	12.94767	11.65457	10.56682	9.64416	8.17550	7.87036
36	13.03521	11.71719	10.61176	9.67651	8.19241	7.88476
37	13.11702	11.77518	10.65299	9.70592	8.20751	7.89757
38	13.19347	11.82887	10.69082	9.73265	8.22099	7.90895
39	13.26493	11.87858	10.72552	9.75696	8.23303	7.91906
40	13.33171	11.92461	10.75736	9.77905	8.24378	7.92806

Annuity Value Factors (in arrears) $\frac{q^n-1}{q^n\cdot(q-1)} = \frac{(1+i)^n-1}{i\cdot(1+i)^n}$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
41	23.41240	20.77204	19.99305	19.25878	17.29437	15.13802
42	23.70136	20.98510	20.18563	19.43288	17.42321	15.22454
43	23.98190	21.19046	20.37079	19.59988	17.54591	15.30617
44	24.25427	21.38839	20.54884	19.76008	17.66277	15.38318
45	24.51871	21.57917	20.72004	19.91375	17.77407	15.45583
46	24.77545	21.76306	20.88465	20.06115	17.88007	15.52437
47	25.02471	21.94030	21.04294	20.20254	17.98102	15.58903
48	25.26671	22.11113	21.19513	20.33817	18.07716	15.65003
49	25.50166	22.27579	21.34147	20.46827	18.16872	15.70757
50	25.72976	22.43449	21.48218	20.59306	18.25593	15.76186
51	25.95123	22.58746	21.61749	20.71277	18.33898	15.81308
52	26.16624	22.73490	21.74758	20.82760	18.41807	15.86139
53	26.37499	22.87702	21.87267	20.93774	18.49340	15.90697
54	26.57766	23.01399	21.99296	21.04340	18.56515	15.94998
55	26.77443	23.14602	22.10861	21.14475	18.63347	15.99054
56	26.96546	23.27327	22.21982	21.24196	18.69854	16.02881
57	27.15094	23.39592	22.32675	21.33522	18.76052	16.06492
58	27.33101	23.51414	22.42957	21.42467	18.81954	16.09898
59	27.50583	23.62809	22.52843	21.51047	18.87575	16.13111
60	27.67556	23.73792	22.62349	21.59278	18.92929	16.16143
61	27.84035	23.84377	22.71489	21.67173	18.98028	16.19003
62	28.00034	23.94581	22.80278	21.74746	19.02883	16.21701
63	28.15567	24.04415	22.88729	21.82011	19.07508	16.24246
64	28.30648	24.13894	22.96855	21.88979	19.11912	16.26647
65	28.45289	24.23030	23.04668	21.95664	19.16107	16.28912
66	28.59504	24.31837	23.12181	22.02075	19.20102	16.31049
67	28.73305	24.40324	23.19405	22.08226	19.23907	16.33065
68	28.86704	24.48505	23.26351	22.14125	19.27530	16.34967
69	28.99712	24.56391	23.33030	22.19785	19.30981	16.36762
70	29.12342	24.63991	23.39451	22.25213	19.34268	16.38454
71	29.24604	24.71317	23.45626	22.30420	19.37398	16.40051
72	29.36509	24.78378	23.51564	22.35415	19.40379	16.41558
73	29.48067	24.85183	23.57273	22.40206	19.43218	16.42979
74	29.59288	24.91743	23.62762	22.44802	19.45922	16.44320
75	29.70183	24.98065	23.68041	22.49211	19.48497	16.45585
76	29.80760	25.04159	23.73116	22.53440	19.5095	16.46778
77	29.91029	25.10033	23.77996	22.57496	19.53285	16.47904
78	30.00999	25.15695	23.82689	22.61387	19.55510	16.48966
79	30.10679	25.21151	23.87201	22.65119	19.57628	16.49968
80	30.20076	25.26411	23.91539	22.68700	19.59646	16.50913

Annuity Value Factors (in arrears) $\frac{q^n-1}{q^n\cdot(q-1)} = \frac{(1+i)^n-1}{i\cdot(1+i)^n}$

n	**i**					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
41	13.39412	11.96723	10.78657	9.79914	8.25337	7.93605
42	13.45245	12.00670	10.81337	9.81740	8.26194	7.94316
43	13.50696	12.04324	10.83795	9.83400	8.26959	7.94947
44	13.55791	12.07707	10.86051	9.84909	8.27642	7.95509
45	13.60552	12.10840	10.88120	9.86281	8.28252	7.96008
46	13.65002	12.13741	10.90018	9.87528	8.28796	7.96451
47	13.69161	12.16427	10.91760	9.88662	8.29282	7.96846
48	13.73047	12.18914	10.93358	9.89693	8.29716	7.97196
49	13.76680	12.21216	10.94823	9.90630	8.30104	7.97508
50	13.80075	12.23348	10.96168	9.91481	8.30450	7.97785
51	13.83247	12.25323	10.97402	9.92256	8.30759	7.98031
52	13.86212	12.27151	10.98534	9.92960	8.31035	7.98250
53	13.88984	12.28843	10.99573	9.93600	8.31281	7.98444
54	13.91573	12.30410	11.00525	9.94182	8.31501	7.98617
55	13.93994	12.31861	11.01399	9.94711	8.31697	7.98771
56	13.96256	12.33205	11.02201	9.95191	8.31872	7.98907
57	13.98370	12.34449	11.02937	9.95629	8.32029	7.99029
58	14.00346	12.35601	11.03612	9.96026	8.32169	7.99137
59	14.02192	12.36668	11.04231	9.96387	8.32294	7.99232
60	14.03918	12.37655	11.04799	9.96716	8.32405	7.99318
61	14.05531	12.38570	11.05320	9.97014	8.32504	7.99394
62	14.07038	12.39416	11.05798	9.97286	8.32593	7.99461
63	14.08447	12.40200	11.06237	9.97532	8.32673	7.99521
64	14.09764	12.40926	11.06640	9.97757	8.32743	7.99574
65	14.10994	12.41598	11.07009	9.97961	8.32807	7.99621
66	14.12144	12.42221	11.07347	9.98146	8.32863	7.99663
67	14.13219	12.42797	11.07658	9.98315	8.32913	7.99701
68	14.14223	12.4333	11.07943	9.98468	8.32958	7.99734
69	14.15162	12.43825	11.08205	9.98607	8.32999	7.99764
70	14.16039	12.44282	11.08445	9.98734	8.33034	7.99790
71	14.16859	12.44706	11.08665	9.98849	8.33066	7.99813
72	14.17625	12.45098	11.08867	9.98954	8.33095	7.99834
73	14.18341	12.45461	11.09052	9.99049	8.33121	7.99852
74	14.19010	12.45797	11.09222	9.99135	8.33143	7.99869
75	14.19636	12.46108	11.09378	9.99214	8.33164	7.99883
76	14.20220	12.46397	11.09521	9.99285	8.33182	7.99896
77	14.20767	12.46664	11.09653	9.99350	8.33198	7.99908
78	14.21277	12.46911	11.09773	9.99409	8.33213	7.99918
79	14.21755	12.47140	11.09883	9.99463	8.33226	7.99927
80	14.22201	12.47351	11.09985	9.99512	8.33237	7.99935

Annuity Value Factors (in arrears) $\frac{q^n-1}{q^n\cdot(q-1)} = \frac{(1+i)^n-1}{i\cdot(1+i)^n}$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
85	30.63115	25.49991	24.10853	22.84527	19.68382	16.54895
90	31.00241	25.69607	24.26728	22.97381	19.75226	16.57870
95	31.32266	25.85925	24.39776	23.07820	19.80589	16.60093
100	31.59891	25.99499	24.50500	23.16297	19.84791	16.61755
105	31.83720	26.10792	24.59315	23.23182	19.88083	16.62996

Annuity Value Factors (in arrears) $\frac{q^n-1}{q^n\cdot(q-1)} = \frac{(1+i)^n-1}{i\cdot(1+i)^n}$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
85	14.24029	12.48197	11.10379	9.99697	8.33279	7.99964
90	14.25333	12.48773	11.10635	9.99812	8.33302	7.99980
95	14.26262	12.49165	11.10802	9.99883	8.33316	7.99989
100	14.26925	12.49432	11.10910	9.99927	8.33323	7.99994
105	14.27398	12.49613	11.10981	9.99955	8.33328	7.99997

Accumulation Factors of Annuity (in advance)

$$\frac{q^n-1}{q-1}\cdot q=\frac{(1+i)^n-1}{i}\cdot(1+i)$$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
1	1.03000	1.03750	1.04000	1.04250	1.05000	1.06000
2	2.09090	2.11391	2.12160	2.12931	2.15250	2.18360
3	3.18363	3.23068	3.24646	3.26230	3.31013	3.37462
4	4.30914	4.38933	4.41632	4.44345	4.52563	4.63709
5	5.46841	5.59143	5.63298	5.67480	5.80191	5.97532
6	6.66246	6.83861	6.89829	6.95848	7.14201	7.39384
7	7.89234	8.13255	8.21423	8.29671	8.54911	8.89747
8	9.15911	9.47503	9.58280	9.69182	10.02656	10.49132
9	10.46388	10.86784	11.00611	11.14622	11.57789	12.18079
10	11.80780	12.31288	12.48635	12.66244	13.20679	13.97164
11	13.19203	13.81212	14.02581	14.24309	14.91713	15.86994
12	14.61779	15.36757	15.62684	15.89092	16.71298	17.88214
13	16.08632	16.98135	17.29191	17.60879	18.59863	20.01507
14	17.59891	18.65565	19.02359	19.39966	20.57856	22.27597
15	19.15688	20.39274	20.82453	21.26665	22.65749	24.67253
16	20.76159	22.19497	22.69751	23.21298	24.84037	27.21288
17	22.41444	24.06478	24.64541	25.24203	27.13238	29.90565
18	24.11687	26.00471	26.67123	27.35732	29.53900	32.75999
19	25.87037	28.01739	28.77808	29.56250	32.06595	35.78559
20	27.67649	30.10554	30.96920	31.86141	34.71925	38.99273
21	29.53678	32.27200	33.24797	34.25802	37.50521	42.39229
22	31.45288	34.51970	35.61789	36.75648	40.43048	45.99583
23	33.42647	36.85168	38.08260	39.36113	43.50200	49.81558
24	35.45926	39.27112	40.64591	42.07648	46.72710	53.86451
25	37.55304	41.78129	43.31174	44.90723	50.11345	58.15638
26	39.70963	44.38559	46.08421	47.85829	53.66913	62.70577
27	41.93092	47.08755	48.96758	50.93477	57.40258	67.52811
28	44.21885	49.89083	51.96629	54.14199	61.32271	72.63980
29	46.57542	52.79924	55.08494	57.48553	65.43885	78.05819
30	49.00268	55.81671	58.32834	60.97116	69.76079	83.80168
31	51.50276	58.94734	61.70147	64.60494	74.29883	89.88978
32	54.07784	62.19536	65.20953	68.39315	79.06377	96.34316
33	56.73018	65.56519	68.85791	72.34236	84.06696	103.1838
34	59.46208	69.06138	72.65222	76.45941	89.32031	110.4348
35	62.27594	72.68868	76.59831	80.75143	94.83632	118.1209
36	65.17422	76.45201	80.70225	85.22587	100.6281	126.2681
37	68.15945	80.35646	84.97034	89.89047	106.7095	134.9042
38	71.23423	84.40733	89.40915	94.75331	113.0950	144.0585
39	74.40126	88.61010	94.02552	99.82283	119.7998	153.7620
40	77.66330	92.97048	98.82654	105.10780	126.8398	164.0477

Accumulation Factors of Annuity (in advance)

$$\mathbf{\frac{q^n-1}{q-1}\cdot q=\frac{(1+i)^n-1}{i}\cdot(1+i)}$$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
1	1.07000	1.08000	1.09000	1.10000	1.12000	1.12500
2	2.21490	2.24640	2.27810	2.31000	2.37440	2.39063
3	3.43994	3.50611	3.57313	3.64100	3.77933	3.81445
4	4.75074	4.86660	4.98471	5.10510	5.35285	5.41626
5	6.15329	6.33593	6.52333	6.71561	7.11519	7.21829
6	7.65402	7.92280	8.20043	8.48717	9.08901	9.24558
7	9.25980	9.63663	10.02847	10.43589	11.29969	11.52628
8	10.97799	11.48756	12.02104	12.57948	13.77566	14.09206
9	12.81645	13.48656	14.19293	14.93742	16.54874	16.97857
10	14.78360	15.64549	16.56029	17.53117	19.65458	20.22589
11	16.88845	17.97713	19.14072	20.38428	23.13313	23.87913
12	19.14064	20.49530	21.95338	23.52271	27.02911	27.98902
13	21.55049	23.21492	25.01919	26.97498	31.39260	32.61264
14	24.12902	26.15211	28.36092	30.77248	36.27971	37.81422
15	26.88805	29.32428	32.00340	34.94973	41.75328	43.66600
16	29.84022	32.75023	35.97370	39.54470	47.88367	50.24925
17	32.99903	36.45024	40.30134	44.59917	54.74971	57.65541
18	36.37896	40.44626	45.01846	50.15909	62.43968	65.98733
19	39.99549	44.76196	50.16012	56.27500	71.05244	75.36075
20	43.86518	49.42292	55.76453	63.00250	80.69874	85.90584
21	48.00574	54.45676	61.87334	70.40275	91.50258	97.76908
22	52.43614	59.89330	68.53194	78.54302	103.6029	111.1152
23	57.17667	65.76476	75.78981	87.49733	117.1552	126.1296
24	62.24904	72.10594	83.70090	97.34706	132.3339	143.0208
25	67.67647	78.95442	92.32398	108.1818	149.3339	162.0234
26	73.48382	86.35077	101.7231	120.0999	168.3740	183.4013
27	79.69769	94.33883	111.9682	133.2099	189.6989	207.4515
28	86.34653	102.9659	123.1354	147.6309	213.5828	234.5079
29	93.46079	112.2832	135.3075	163.4940	240.3327	264.9464
30	101.0730	122.3459	148.5752	180.9434	270.2926	299.1897
31	109.2182	133.2135	163.0370	200.1378	303.8477	337.7135
32	117.9334	144.9506	178.8003	221.2515	341.4294	381.0526
33	127.2588	157.6267	195.9823	244.4767	383.5210	429.8092
34	137.2369	171.3168	214.7108	270.0244	430.6635	484.6604
35	147.9135	186.1021	235.1247	298.1268	483.4631	546.3679
36	159.3374	202.0703	257.3759	329.0395	542.5987	615.7889
37	171.5610	219.3159	281.6298	363.0434	608.8305	693.8875
38	184.6403	237.9412	308.0665	400.4478	683.0102	781.7485
39	198.6351	258.0565	336.8824	441.5926	766.0914	880.5920
40	213.6096	279.7810	368.2919	486.8518	859.1424	991.7910

Accumulation Factors of Annuity (in advance)

$$\frac{q^n-1}{q-1}\cdot q=\frac{(1+i)^n-1}{i}\cdot(1+i)$$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
41	81.02320	97.49437	103.8196	110.6174	134.2318	174.9505
42	84.48389	102.1879	109.0124	116.3611	141.9933	186.5076
43	88.04841	107.0575	114.4129	122.3490	150.1430	198.7580
44	91.71986	112.1096	120.0294	128.5913	158.7002	211.7435
45	95.50146	117.3512	125.8706	135.0989	167.6852	225.5081
46	99.39650	122.7894	131.9454	141.8831	177.1194	240.0986
47	103.4084	128.4315	138.2632	148.9557	187.0254	255.5645
48	107.5406	134.2852	144.8337	156.3288	197.4267	271.9584
49	111.7969	140.3584	151.6671	164.0153	208.3480	289.3359
50	116.1808	146.6593	158.7738	172.0284	219.8154	307.7561
51	120.6962	153.1965	166.1647	180.3821	231.8562	327.2814
52	125.3471	159.9789	173.8513	189.0909	244.4990	347.9783
53	130.1375	167.0156	181.8454	198.1697	257.7739	369.9170
54	135.0716	174.3162	190.1592	207.6344	271.7126	393.1720
55	140.1538	181.8906	198.8055	217.5014	286.3482	417.8223
56	145.3884	189.7490	207.7978	227.7877	301.7157	443.9517
57	150.7800	197.9020	217.1497	238.5112	317.8514	471.6488
58	156.3334	206.3609	226.8757	249.6904	334.7940	501.0077
59	162.0534	215.1369	236.9907	261.3447	352.5837	532.1282
60	167.9450	224.2420	247.5103	273.4944	371.2629	565.1159
61	174.0134	233.6886	258.4507	286.1604	390.8760	600.0828
62	180.2638	243.4894	269.8288	299.3647	411.4699	637.1478
63	186.7017	253.6578	281.6619	313.1302	433.0933	676.4367
64	193.3328	264.2074	293.9684	327.4808	455.7980	718.0829
65	200.1627	275.1527	306.7671	342.4412	479.6379	762.2278
66	207.1976	286.5085	320.0778	358.0374	504.6698	809.0215
67	214.4436	298.2900	333.9209	374.2965	530.9533	858.6228
68	221.9069	310.5134	348.3177	391.2466	558.5510	911.2002
69	229.5941	323.1952	363.2905	408.9171	587.5285	966.9322
70	237.5119	336.3525	378.8621	427.3386	617.9549	1026.008
71	245.6672	350.0032	395.0566	446.5430	649.9027	1088.629
72	254.0673	364.1658	411.8988	466.5636	683.4478	1155.006
73	262.7193	378.8595	429.4148	487.4350	718.6702	1225.367
74	271.6309	394.1043	447.6314	509.1935	755.6537	1299.949
75	280.8098	409.9207	466.5766	531.8767	794.4864	1379.006
76	290.2641	426.3302	486.2797	555.5240	835.2607	1462.806
77	300.0020	443.3551	506.7709	580.1762	878.0738	1551.634
78	310.0321	461.0184	528.0817	605.8762	923.0274	1645.792
79	320.3630	479.3441	550.2450	632.6685	970.2288	1745.600
80	331.0039	498.3570	573.2948	660.5994	1019.790	1851.396

Accumulation Factors of Annuity (in advance)

$$\mathbf{\frac{q^n-1}{q-1}\cdot q=\frac{(1+i)^n-1}{i}\cdot(1+i)}$$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
41	229.6322	303.2435	402.5281	536.6370	963.3595	1116.890
42	246.7765	328.5830	439.8457	591.4007	1080.083	1257.626
43	265.1209	355.9496	480.5218	651.6408	1210.813	1415.954
44	284.7493	385.5056	524.8587	717.9048	1357.230	1594.074
45	305.7518	417.4261	573.1860	790.7953	1521.218	1794.458
46	328.2244	451.9002	625.8628	870.9749	1704.884	2019.890
47	352.2701	489.1322	683.2804	959.1723	1910.590	2273.501
48	377.9990	529.3427	745.8656	1056.190	2140.981	2558.814
49	405.5289	572.7702	814.0836	1162.909	2399.018	2879.791
50	434.9860	619.6718	888.4411	1280.299	2688.020	3240.890
51	466.5050	670.3255	969.4908	1409.429	3011.703	3647.126
52	500.2303	725.0316	1057.835	1551.472	3374.227	4104.142
53	536.3164	784.1141	1154.130	1707.719	3780.255	4618.284
54	574.9286	847.9232	1259.092	1879.591	4235.005	5196.695
55	616.2436	916.8371	1373.500	2068.651	4744.326	5847.407
56	660.4506	991.2640	1498.205	2276.616	5314.765	6579.458
57	707.7522	1071.645	1634.134	2505.377	5953.656	7403.015
58	758.3648	1158.457	1782.296	2757.015	6669.215	8329.517
59	812.5204	1252.213	1943.792	3033.816	7470.641	9371.831
60	870.4668	1353.470	2119.823	3338.298	8368.238	10544.44
61	932.4695	1462.828	2311.698	3673.228	9373.547	11863.61
62	998.8124	1580.934	2520.840	4041.651	10499.49	13347.69
63	1069.799	1708.489	2748.806	4446.916	11760.55	15017.28
64	1145.755	1846.248	2997.288	4892.707	13172.94	16895.56
65	1227.028	1995.028	3268.134	5383.078	14754.81	19008.63
66	1313.990	2155.710	3563.357	5922.486	16526.51	21385.84
67	1407.039	2329.247	3885.149	6515.834	18510.81	24060.19
68	1506.602	2516.667	4235.902	7168.518	20733.22	27068.84
69	1613.134	2719.080	4618.223	7886.470	23222.33	30453.57
70	1727.124	2937.686	5034.953	8676.217	26010.13	34261.39
71	1849.092	3173.781	5489.189	9544.938	29132.47	38545.19
72	1979.599	3428.764	5984.306	10500.53	32629.48	43364.47
73	2119.241	3704.145	6523.984	11551.69	36546.14	48786.15
74	2268.657	4001.557	7112.232	12707.95	40932.80	54885.54
75	2428.533	4322.761	7753.423	13979.85	45845.85	61747.36
76	2599.601	4669.662	8452.321	15378.93	51348.48	69466.91
77	2782.643	5044.315	9214.120	16917.93	57511.41	78151.39
78	2978.498	5448.940	10044.48	18610.82	64413.90	87921.44
79	3188.063	5885.935	10949.57	20473.00	72144.69	98912.75
80	3412.297	6357.890	11936.13	22521.40	80803.18	111278.0

Accumulation Factors of Annuity (in advance)

$$\frac{q^n-1}{q-1}\cdot q=\frac{(1+i)^n-1}{i}\cdot(1+i)$$

n	i					
	0.03	0.0375	0.04	0.0425	0.05	0.06
85	389.1927	604.6663	703.1337	819.1016	1307.341	2483.561
90	456.6494	732.4606	861.1027	1014.273	1674.338	3329.540
95	534.8502	886.0822	1053.296	1254.596	2142.728	4461.651
100	625.5064	1070.751	1287.129	1550.518	2740.526	5976.670
105	730.6017	1292.741	1571.622	1914.899	3503.485	8004.108

Accumulation Factors of Annuity (in advance)

$$\frac{q^n-1}{q-1}\cdot q=\frac{(1+i)^n-1}{i}\cdot(1+i)$$

n	i					
	0.07	0.08	0.09	0.10	0.12	0.125
85	4792.076	9348.163	18371.73	36277.66	142409.9	200533.7
90	6727.288	13741.85	28273.71	58432.25	250982.1	361375.5
95	9441.523	20197.63	43509.13	94112.44	442323.2	651217.6
100	13248.38	29683.28	66950.72	151575.7	779531.8	1173522.5
105	18587.69	43620.81	103018.5	244121.0	1373809	2114732.9

Accumulation Factors of Annuity (in arrears) $\frac{q^n-1}{q-1} = \frac{(1+i)^n-1}{i}$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0300	2.0375	2.0400	2.0425	2.0500	2.0600
3	3.0909	3.1139	3.1216	3.1293	3.1525	3.1836
4	4.1836	4.2307	4.2465	4.2623	4.3101	4.3746
5	5.3091	5.3893	5.4163	5.4434	5.5256	5.6371
6	6.4684	6.5914	6.6330	6.6748	6.8019	6.9753
7	7.6625	7.8386	7.8983	7.9585	8.1420	8.3938
8	8.8923	9.1326	9.2142	9.2967	9.5491	9.8975
9	10.1591	10.4750	10.5828	10.6918	11.0266	11.4913
10	11.4639	11.8678	12.0061	12.1462	12.5779	13.1808
11	12.8078	13.3129	13.4864	13.6624	14.2068	14.9716
12	14.1920	14.8121	15.0258	15.2431	15.9171	16.8699
13	15.6178	16.3676	16.6268	16.8909	17.7130	18.8821
14	17.0863	17.9814	18.2919	18.6088	19.5986	21.0151
15	18.5989	19.6557	20.0236	20.3997	21.5786	23.2760
16	20.1569	21.3927	21.8245	22.2666	23.6575	25.6725
17	21.7616	23.1950	23.6975	24.2130	25.8404	28.2129
18	23.4144	25.0648	25.6454	26.2420	28.1324	30.9057
19	25.1169	27.0047	27.6712	28.3573	30.5390	33.7600
20	26.8704	29.0174	29.7781	30.5625	33.0660	36.7856
21	28.6765	31.1055	31.9692	32.8614	35.7193	39.9927
22	30.5368	33.2720	34.2480	35.2580	38.5052	43.3923
23	32.4529	35.5197	36.6179	37.7565	41.4305	46.9958
24	34.4265	37.8517	39.0826	40.3611	44.5020	50.8156
25	36.4593	40.2711	41.6459	43.0765	47.7271	54.8645
26	38.5530	42.7813	44.3117	45.9072	51.1135	59.1564
27	40.7096	45.3856	47.0842	48.8583	54.6691	63.7058
28	42.9309	48.0875	49.9676	51.9348	58.4026	68.5281
29	45.2189	50.8908	52.9663	55.1420	62.3227	73.6398
30	47.5754	53.7992	56.0849	58.4855	66.4388	79.0582
31	50.0027	56.8167	59.3283	61.9712	70.7608	84.8017
32	52.5028	59.9473	62.7015	65.6049	75.2988	90.8898
33	55.0778	63.1954	66.2095	69.3931	80.0638	97.3432
34	57.7302	66.5652	69.8579	73.3424	85.0670	104.1838
35	60.4621	70.0614	73.6522	77.4594	90.3203	111.4348
36	63.2759	73.6887	77.5983	81.7514	95.8363	119.1209
37	66.1742	77.4520	81.7022	86.2259	101.6281	127.2681
38	69.1594	81.3565	85.9703	90.8905	107.7095	135.9042
39	72.2342	85.4073	90.4091	95.7533	114.0950	145.0585
40	75.4013	89.6101	95.0255	100.8228	120.7998	154.7620

Accumulation Factors of Annuity (in arrears) $\frac{\mathbf{q^n-1}}{\mathbf{q-1}} = \frac{\mathbf{(1+i)^n-1}}{\mathbf{i}}$

n	**i**					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
1	1.000	1.000	1.000	1.000	1.000	1.000
2	2.0700	2.0800	2.0900	2.100	2.1200	2.1250
3	3.2149	3.2464	3.2781	3.3100	3.3744	3.3906
4	4.4399	4.5061	4.5731	4.6410	4.7793	4.8145
5	5.7507	5.8666	5.9847	6.1051	6.3528	6.4163
6	7.1533	7.3359	7.5233	7.7156	8.1152	8.2183
7	8.6540	8.9228	9.2004	9.4872	10.0890	10.2456
8	10.2598	10.6366	11.0285	11.4359	12.2997	12.5263
9	11.9780	12.4876	13.0210	13.5795	14.7757	15.0921
10	13.8164	14.4866	15.1929	15.9374	17.5487	17.9786
11	15.7836	16.6455	17.5603	18.5312	20.6546	21.2259
12	17.8885	18.9771	20.1407	21.3843	24.1331	24.8791
13	20.1406	21.4953	22.9534	24.5227	28.0291	28.9890
14	22.5505	24.2149	26.0192	27.9750	32.3926	33.6126
15	25.1290	27.1521	29.3609	31.7725	37.2797	38.8142
16	27.8881	30.3243	33.0034	35.9497	42.7533	44.6660
17	30.8402	33.7502	36.9737	40.5447	48.8837	51.2493
18	33.9990	37.4502	41.3013	45.5992	55.7497	58.6554
19	37.3790	41.4463	46.0185	51.1591	63.4397	66.9873
20	40.9955	45.7620	51.1601	57.2750	72.0524	76.3608
21	44.8652	50.4229	56.7645	64.0025	81.6987	86.9058
22	49.0057	55.4568	62.8733	71.4027	92.5026	98.7691
23	53.4361	60.8933	69.5319	79.5430	104.6029	112.1152
24	58.1767	66.7648	76.7898	88.4973	118.1552	127.1296
25	63.2490	73.1059	84.7009	98.3471	133.3339	144.0208
26	68.6765	79.9544	93.3240	109.1818	150.3339	163.0234
27	74.4838	87.3508	102.7231	121.0999	169.3740	184.4013
28	80.6977	95.3388	112.9682	134.2099	190.6989	208.4515
29	87.3465	103.9659	124.1354	148.6309	214.5828	235.5079
30	94.4608	113.2832	136.3075	164.4940	241.3327	265.9464
31	102.0730	123.3459	149.5752	181.9434	271.2926	300.1897
32	110.2182	134.2135	164.0370	201.1378	304.8477	338.7135
33	118.9334	145.9506	179.8003	222.2515	342.4294	382.0526
34	128.2588	158.6267	196.9823	245.4767	384.5210	430.8092
35	138.2369	172.3168	215.7108	271.0244	431.6635	485.6604
36	148.9135	187.1021	236.1247	299.1268	484.4631	547.3679
37	160.3374	203.0703	258.3759	330.0395	543.5987	616.7889
38	172.5610	220.3159	282.6298	364.0434	609.8305	694.8875
39	185.6403	238.9412	309.0665	401.4478	684.0102	782.7485
40	199.6351	259.0565	337.8824	442.5926	767.0914	881.5920

Accumulation Factors of Annuity (in arrears) $\frac{q^n-1}{q-1} = \frac{(1+i)^n-1}{i}$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
41	78.6633	93.9705	99.8265	106.1078	127.8398	165.0477
42	82.0232	98.4944	104.8196	111.6174	135.2318	175.9505
43	85.4839	103.1879	110.0124	117.3611	142.9933	187.5076
44	89.0484	108.0575	115.4129	123.3490	151.1430	199.7580
45	92.7199	113.1096	121.0294	129.5913	159.7002	212.7435
46	96.5015	118.3512	126.8706	136.0989	168.6852	226.5081
47	100.3965	123.7894	132.9454	142.8831	178.1194	241.0986
48	104.4084	129.4315	139.2632	149.9557	188.0254	256.5645
49	108.5406	135.2852	145.8337	157.3288	198.4267	272.9584
50	112.7969	141.3584	152.6671	165.0153	209.3480	290.3359
51	117.1808	147.6593	159.7738	173.0284	220.8154	308.7561
52	121.6962	154.1965	167.1647	181.3821	232.8562	328.2814
53	126.3471	160.9789	174.8513	190.0909	245.4990	348.9783
54	131.1375	168.0156	182.8454	199.1697	258.7739	370.9170
55	136.0716	175.3162	191.1592	208.6344	272.7126	394.1720
56	141.1538	182.8906	199.8055	218.5014	287.3482	418.8223
57	146.3884	190.7490	208.7978	228.7877	302.7157	444.9517
58	151.7800	198.9020	218.1497	239.5112	318.8514	472.6488
59	157.3334	207.3609	227.8757	250.6904	335.7940	502.0077
60	163.0534	216.1369	237.9907	262.3447	353.5837	533.1282
61	168.9450	225.2420	248.5103	274.4944	372.2629	566.1159
62	175.0134	234.6886	259.4507	287.1604	391.8760	601.0828
63	181.2638	244.4894	270.8288	300.3647	412.4699	638.1478
64	187.7017	254.6578	282.6619	314.1302	434.0933	677.4367
65	194.3328	265.2074	294.9684	328.4808	456.7980	719.0829
66	201.1627	276.1527	307.7671	343.4412	480.6379	763.2278
67	208.1976	287.5085	321.0778	359.0374	505.6698	810.0215
68	215.4436	299.2900	334.9209	375.2965	531.9533	859.6228
69	222.9069	311.5134	349.3177	392.2466	559.5510	912.2002
70	230.5941	324.1952	364.2905	409.9171	588.5285	967.9322
71	238.5119	337.3525	379.8621	428.3386	618.9549	1027.0081
72	246.6672	351.0032	396.0566	447.5430	650.9027	1089.6286
73	255.0673	365.1658	412.8988	467.5636	684.4478	1156.0063
74	263.7193	379.8595	430.4148	488.4350	719.6702	1226.3667
75	272.6309	395.1043	448.6314	510.1935	756.6537	1300.9487
76	281.8098	410.9207	467.5766	532.8767	795.4864	1380.0056
77	291.2641	427.3302	487.2797	556.5240	836.2607	1463.8059
78	301.0020	444.3551	507.7709	581.1762	879.0738	1552.6343
79	311.0321	462.0184	529.0817	606.8762	924.0274	1646.7924
80	321.3630	480.3441	551.2450	633.6685	971.2288	1746.5999

Accumulation Factors of Annuity (in arrears) $\frac{q^n-1}{q-1} = \frac{(1+i)^n-1}{i}$

n	i					
	0.07	0.08	0.09	0.10	0.12	0.125
41	214.61	280.78	369.29	487.85	860.14	992.79
42	230.63	304.24	403.53	537.64	964.36	1117.89
43	247.78	329.58	440.85	592.40	1081.08	1258.63
44	266.12	356.95	481.52	652.64	1211.81	1416.95
45	285.75	386.51	525.86	718.90	1358.23	1595.07
46	306.75	418.43	574.19	791.80	1522.22	1795.46
47	329.22	452.90	626.86	871.97	1705.88	2020.89
48	353.27	490.13	684.28	960.17	1911.59	2274.50
49	379.00	530.34	746.87	1057.19	2141.98	2559.81
50	406.53	573.77	815.08	1163.91	2400.02	2880.79
51	435.99	620.67	889.44	1281.30	2689.02	3241.89
52	467.50	671.33	970.49	1410.43	3012.70	3648.13
53	501.23	726.03	1058.83	1552.47	3375.23	4105.14
54	537.32	785.11	1155.13	1708.72	3781.25	4619.28
55	575.93	848.92	1260.09	1880.59	4236.01	5197.70
56	617.24	917.84	1374.50	2069.65	4745.33	5848.41
57	661.45	992.26	1499.21	2277.62	5315.76	6580.46
58	708.75	1072.65	1635.13	2506.38	5954.66	7404.01
59	759.36	1159.46	1783.30	2758.01	6670.22	8330.52
60	813.52	1253.21	1944.79	3034.82	7471.64	9372.83
61	871.47	1354.47	2120.82	3339.30	8369.24	10545.44
62	933.47	1463.83	2312.70	3674.23	9374.55	11864.61
63	999.81	1581.93	2521.84	4042.65	10500.49	13348.69
64	1070.80	1709.49	2749.81	4447.92	11761.55	15018.28
65	1146.76	1847.25	2998.29	4893.71	13173.94	16896.56
66	1228.03	1996.03	3269.13	5384.08	14755.81	19009.63
67	1314.99	2156.71	3564.36	5923.49	16527.51	21386.84
68	1408.04	2330.25	3886.15	6516.83	18511.81	24061.19
69	1507.60	2517.67	4236.90	7169.52	20734.22	27069.84
70	1614.13	2720.08	4619.22	7887.47	23223.33	30454.57
71	1728.12	2938.69	5035.95	8677.22	26011.13	34262.39
72	1850.09	3174.78	5490.19	9545.94	29133.47	38546.19
73	1980.60	3429.76	5985.31	10501.53	32630.48	43365.47
74	2120.24	3705.15	6524.98	11552.69	36547.14	48787.15
75	2269.66	4002.56	7113.23	12708.95	40933.80	54886.54
76	2429.53	4323.76	7754.42	13980.85	45846.85	61748.36
77	2600.60	4670.66	8453.32	15379.93	51349.48	69467.91
78	2783.64	5045.32	9215.12	16918.93	57512.41	78152.39
79	2979.50	5449.94	10045.48	18611.82	64414.90	87922.44
80	3189.06	5886.94	10950.57	20474.00	72145.69	98913.75

Accumulation Factors of Annuity (in arrears) $\frac{q^n-1}{q-1} = \frac{(1+i)^n-1}{i}$

n	**i**					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
85	377.8570	582.8109	676.0901	785.7090	1245.0871	2342.9817
90	443.3489	705.9861	827.9833	972.9235	1594.6073	3141.0752
95	519.2720	854.0551	1012.7846	1203.4496	2040.6935	4209.1042
100	607.2877	1032.0488	1237.6237	1487.3070	2610.0252	5638.3681
105	709.3221	1246.0150	1511.1748	1836.8338	3336.6526	7551.0454

Accumulation Factors of Annuity (in arrears) $\frac{q^n-1}{q-1} = \frac{(1+i)^n-1}{i}$

n	**i**					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
85	4478.58	8655.71	16854.80	32979.69	127151.71	178252.20
90	6287.19	12723.94	25939.18	53120.23	224091.12	321222.67
95	8823.85	18701.51	39916.63	85556.76	394931.47	578860.10
100	12381.66	27484.52	61422.68	137796.12	696010.55	1043131.12
105	17371.67	40389.64	94512.38	221928.14	1226614.75	1879762.56

Annuity Factors (in advance) $\frac{q^n(q-1)}{q^n-1}\cdot q^n = \frac{i\cdot(1+i)^n}{(1+i)^n-1}\cdot\frac{1}{(1+i)^n}$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.49261	0.49080	0.49020	0.48960	0.48780	0.48544
3	0.32353	0.32114	0.32035	0.31956	0.31721	0.31411
4	0.23903	0.23637	0.23549	0.23462	0.23201	0.22859
5	0.18835	0.18555	0.18463	0.18371	0.18097	0.17740
6	0.15460	0.15171	0.15076	0.14982	0.14702	0.14336
7	0.13051	0.12757	0.12661	0.12565	0.12282	0.11914
8	0.11246	0.10950	0.10853	0.10756	0.10472	0.10104
9	0.09843	0.09547	0.09449	0.09353	0.09069	0.08702
10	0.08723	0.08426	0.08329	0.08233	0.07950	0.07587
11	0.07808	0.07512	0.07415	0.07319	0.07039	0.06679
12	0.07046	0.06751	0.06655	0.06560	0.06283	0.05928
13	0.06403	0.06110	0.06014	0.05920	0.05646	0.05296
14	0.05853	0.05561	0.05467	0.05374	0.05102	0.04758
15	0.05377	0.05088	0.04994	0.04902	0.04634	0.04296
16	0.04961	0.04674	0.04582	0.04491	0.04227	0.03895
17	0.04595	0.04311	0.04220	0.04130	0.03870	0.03544
18	0.04271	0.03990	0.03899	0.03811	0.03555	0.03236
19	0.03981	0.03703	0.03614	0.03526	0.03275	0.02962
20	0.03722	0.03446	0.03358	0.03272	0.03024	0.02718
21	0.03487	0.03215	0.03128	0.03043	0.02800	0.02500
22	0.03275	0.03006	0.02920	0.02836	0.02597	0.02305
23	0.03081	0.02815	0.02731	0.02649	0.02414	0.02128
24	0.02905	0.02642	0.02559	0.02478	0.02247	0.01968
25	0.02743	0.02483	0.02401	0.02321	0.02095	0.01823
26	0.02594	0.02337	0.02257	0.02178	0.01956	0.01690
27	0.02456	0.02203	0.02124	0.02047	0.01829	0.01570
28	0.02329	0.02080	0.02001	0.01925	0.01712	0.01459
29	0.02211	0.01965	0.01888	0.01813	0.01605	0.01358
30	0.02102	0.01859	0.01783	0.01710	0.01505	0.01265
31	0.02000	0.01760	0.01686	0.01614	0.01413	0.01179
32	0.01905	0.01668	0.01595	0.01524	0.01328	0.01100
33	0.01816	0.01582	0.01510	0.01441	0.01249	0.01027
34	0.01732	0.01502	0.01431	0.01363	0.01176	0.00960
35	0.01654	0.01427	0.01358	0.01291	0.01107	0.00897
36	0.01580	0.01357	0.01289	0.01223	0.01043	0.00839
37	0.01511	0.01291	0.01224	0.01160	0.00984	0.00786
38	0.01446	0.01229	0.01163	0.01100	0.00928	0.00736
39	0.01384	0.01171	0.01106	0.01044	0.00876	0.00689
40	0.01326	0.01116	0.01052	0.00992	0.00828	0.00646

Annuity Factors (in advance) $\frac{q^n(q-1)}{q^n-1}\cdot q^n = \frac{i\cdot(1+i)^n}{(1+i)^n-1}\cdot\frac{1}{(1+i)^n}$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.48309	0.48077	0.47847	0.47619	0.47170	0.47059
3	0.31105	0.30803	0.30505	0.30211	0.29635	0.29493
4	0.22523	0.22192	0.21867	0.21547	0.20923	0.20771
5	0.17389	0.17046	0.16709	0.16380	0.15741	0.15585
6	0.13980	0.13632	0.13292	0.12961	0.12323	0.12168
7	0.11555	0.11207	0.10869	0.10541	0.09912	0.09760
8	0.09747	0.09401	0.09067	0.08744	0.08130	0.07983
9	0.08349	0.08008	0.07680	0.07364	0.06768	0.06626
10	0.07238	0.06903	0.06582	0.06275	0.05698	0.05562
11	0.06336	0.06008	0.05695	0.05396	0.04842	0.04711
12	0.05590	0.05270	0.04965	0.04676	0.04144	0.04019
13	0.04965	0.04652	0.04357	0.04078	0.03568	0.03450
14	0.04434	0.04130	0.03843	0.03575	0.03087	0.02975
15	0.03979	0.03683	0.03406	0.03147	0.02682	0.02576
16	0.03586	0.03298	0.03030	0.02782	0.02339	0.02239
17	0.03243	0.02963	0.02705	0.02466	0.02046	0.01951
18	0.02941	0.02670	0.02421	0.02193	0.01794	0.01705
19	0.02675	0.02413	0.02173	0.01955	0.01576	0.01493
20	0.02439	0.02185	0.01955	0.01746	0.01388	0.01310
21	0.02229	0.01983	0.01762	0.01562	0.01224	0.01151
22	0.02041	0.01803	0.01590	0.01401	0.01081	0.01012
23	0.01871	0.01642	0.01438	0.01257	0.00956	0.00892
24	0.01719	0.01498	0.01302	0.01130	0.00846	0.00787
25	0.01581	0.01368	0.01181	0.01017	0.00750	0.00694
26	0.01456	0.01251	0.01072	0.00916	0.00665	0.00613
27	0.01343	0.01145	0.00973	0.00826	0.00590	0.00542
28	0.01239	0.01049	0.00885	0.00745	0.00524	0.00480
29	0.01145	0.00962	0.00806	0.00673	0.00466	0.00425
30	0.01059	0.00883	0.00734	0.00608	0.00414	0.00376
31	0.00980	0.00811	0.00669	0.00550	0.00369	0.00333
32	0.00907	0.00745	0.00610	0.00497	0.00328	0.00295
33	0.00841	0.00685	0.00556	0.00450	0.00292	0.00262
34	0.00780	0.00630	0.00508	0.00407	0.00260	0.00232
35	0.00723	0.00580	0.00464	0.00369	0.00232	0.00206
36	0.00672	0.00534	0.00424	0.00334	0.00206	0.00183
37	0.00624	0.00492	0.00387	0.00303	0.00184	0.00162
38	0.00580	0.00454	0.00354	0.00275	0.00164	0.00144
39	0.00539	0.00419	0.00324	0.00249	0.00146	0.00128
40	0.00501	0.00386	0.00296	0.00226	0.00130	0.00113

Annuity Factors (in advance) $\frac{q^n(q-1)}{q^n-1} \cdot q^n = \frac{i\cdot(1+i)^n}{(1+i)^n-1} \cdot \frac{1}{(1+i)^n}$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
41	0.01271	0.01064	0.01002	0.00942	0.00782	0.00606
42	0.01219	0.01015	0.00954	0.00896	0.00739	0.00568
43	0.01170	0.00969	0.00909	0.00852	0.00699	0.00533
44	0.01123	0.00925	0.00866	0.00811	0.00662	0.00501
45	0.01079	0.00884	0.00826	0.00772	0.00626	0.00470
46	0.01036	0.00845	0.00788	0.00735	0.00593	0.00441
47	0.00996	0.00808	0.00752	0.00700	0.00561	0.00415
48	0.00958	0.00773	0.00718	0.00667	0.00532	0.00390
49	0.00921	0.00739	0.00686	0.00636	0.00504	0.00366
50	0.00887	0.00707	0.00655	0.00606	0.00478	0.00344
51	0.00853	0.00677	0.00626	0.00578	0.00453	0.00324
52	0.00822	0.00649	0.00598	0.00551	0.00429	0.00305
53	0.00791	0.00621	0.00572	0.00526	0.00407	0.00287
54	0.00763	0.00595	0.00547	0.00502	0.00386	0.00270
55	0.00735	0.00570	0.00523	0.00479	0.00367	0.00254
56	0.00708	0.00547	0.00500	0.00458	0.00348	0.00239
57	0.00683	0.00524	0.00479	0.00437	0.00330	0.00225
58	0.00659	0.00503	0.00458	0.00418	0.00314	0.00212
59	0.00636	0.00482	0.00439	0.00399	0.00298	0.00199
60	0.00613	0.00463	0.00420	0.00381	0.00283	0.00188
61	0.00592	0.00444	0.00402	0.00364	0.00269	0.00177
62	0.00571	0.00426	0.00385	0.00348	0.00255	0.00166
63	0.00552	0.00409	0.00369	0.00333	0.00242	0.00157
64	0.00533	0.00393	0.00354	0.00318	0.00230	0.00148
65	0.00515	0.00377	0.00339	0.00304	0.00219	0.00139
66	0.00497	0.00362	0.00325	0.00291	0.00208	0.00131
67	0.00480	0.00348	0.00311	0.00279	0.00198	0.00123
68	0.00464	0.00334	0.00299	0.00266	0.00188	0.00116
69	0.00449	0.00321	0.00286	0.00255	0.00179	0.00110
70	0.00434	0.00308	0.00275	0.00244	0.00170	0.00103
71	0.00419	0.00296	0.00263	0.00233	0.00162	0.00097
72	0.00405	0.00285	0.00252	0.00223	0.00154	0.00092
73	0.00392	0.00274	0.00242	0.00214	0.00146	0.00087
74	0.00379	0.00263	0.00232	0.00205	0.00139	0.00082
75	0.00367	0.00253	0.00223	0.00196	0.00132	0.00077
76	0.00355	0.00243	0.00214	0.00188	0.00126	0.00072
77	0.00343	0.00234	0.00205	0.00180	0.00120	0.00068
78	0.00332	0.00225	0.00197	0.00172	0.00114	0.00064
79	0.00322	0.00216	0.00189	0.00165	0.00108	0.00061
80	0.00311	0.00208	0.00181	0.00158	0.00103	0.00057

Annuity Factors (in advance) $\frac{q^n(q-1)}{q^n-1}\cdot q^n = \frac{i\cdot(1+i)^n}{(1+i)^n-1}\cdot\frac{1}{(1+i)^n}$

n	i					
	0.07	0.08	0.09	0.10	0.12	0.125
41	0.00466	0.00356	0.00271	0.00205	0.00116	0.00101
42	0.00434	0.00329	0.00248	0.00186	0.00104	0.00089
43	0.00404	0.00303	0.00227	0.00169	0.00092	0.00079
44	0.00376	0.00280	0.00208	0.00153	0.00083	0.00071
45	0.00350	0.00259	0.00190	0.00139	0.00074	0.00063
46	0.00326	0.00239	0.00174	0.00126	0.00066	0.00056
47	0.00304	0.00221	0.00160	0.00115	0.00059	0.00049
48	0.00283	0.00204	0.00146	0.00104	0.00052	0.00044
49	0.00264	0.00189	0.00134	0.00095	0.00047	0.00039
50	0.00246	0.00174	0.00123	0.00086	0.00042	0.00035
51	0.00229	0.00161	0.00112	0.00078	0.00037	0.00031
52	0.00214	0.00149	0.00103	0.00071	0.00033	0.00027
53	0.00200	0.00138	0.00094	0.00064	0.00030	0.00024
54	0.00186	0.00127	0.00087	0.00059	0.00026	0.00022
55	0.00174	0.00118	0.00079	0.00053	0.00024	0.00019
56	0.00162	0.00109	0.00073	0.00048	0.00021	0.00017
57	0.00151	0.00101	0.00067	0.00044	0.00019	0.00015
58	0.00141	0.00093	0.00061	0.00040	0.00017	0.00014
59	0.00132	0.00086	0.00056	0.00036	0.00015	0.00012
60	0.00123	0.00080	0.00051	0.00033	0.00013	0.00011
61	0.00115	0.00074	0.00047	0.00030	0.00012	0.00009
62	0.00107	0.00068	0.00043	0.00027	0.00011	0.00008
63	0.00100	0.00063	0.00040	0.00025	0.00010	0.00007
64	0.00093	0.00058	0.00036	0.00022	0.00009	0.00007
65	0.00087	0.00054	0.00033	0.00020	0.00008	0.00006
66	0.00081	0.00050	0.00031	0.00019	0.00007	0.00005
67	0.00076	0.00046	0.00028	0.00017	0.00006	0.00005
68	0.00071	0.00043	0.00026	0.00015	0.00005	0.00004
69	0.00066	0.00040	0.00024	0.00014	0.00005	0.00004
70	0.00062	0.00037	0.00022	0.00013	0.00004	0.00003
71	0.00058	0.00034	0.00020	0.00012	0.00004	0.00003
72	0.00054	0.00031	0.00018	0.00010	0.00003	0.00003
73	0.00050	0.00029	0.00017	0.00010	0.00003	0.00002
74	0.00047	0.00027	0.00015	0.00009	0.00003	0.00002
75	0.00044	0.00025	0.00014	0.00008	0.00002	0.00002
76	0.00041	0.00023	0.00013	0.00007	0.00002	0.00002
77	0.00038	0.00021	0.00012	0.00007	0.00002	0.00001
78	0.00036	0.00020	0.00011	0.00006	0.00002	0.00001
79	0.00034	0.00018	0.00010	0.00005	0.00002	0.00001
80	0.00031	0.00017	0.00009	0.00005	0.00001	0.00001

Annuity Factors (in advance) $\frac{q^n(q-1)}{q^n-1} \cdot q^n = \frac{i \cdot (1+i)^n}{(1+i)^n-1} \cdot \frac{1}{(1+i)^n}$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
85	0.00265	0.00172	0.00148	0.00127	0.00080	0.00043
90	0.00226	0.00142	0.00121	0.00103	0.00063	0.00032
95	0.00193	0.00117	0.00099	0.00083	0.00049	0.00024
100	0.00165	0.00097	0.00081	0.00067	0.00038	0.00018
105	0.00141	0.00080	0.00066	0.00054	0.00030	0.00013

Annuity Factors (in advance) $\frac{q^n(q-1)}{q^n-1} \cdot q^n = \frac{i \cdot (1+i)^n}{(1+i)^n-1} \cdot \frac{1}{(1+i)^n}$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
85	0.00022	0.00012	0.00006	0.00003	0.00001	0.00001
90	0.00016	0.00008	0.00004	0.00002	0.00000	0.00000
95	0.00011	0.00005	0.00003	0.00001	0.00000	0.00000
100	0.00008	0.00004	0.00002	0.00001	0.00000	0.00000
105	0.00006	0.00002	0.00001	0.00000	0.00000	0.00000

Annuity Factors (in arrears) $\frac{\mathbf{q^n(q-1)}}{\mathbf{q^n-1}} = \frac{\mathbf{i\cdot(1+i)^n}}{\mathbf{(1+i)^n-1}}$

n	**i**					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
1	1.03000	1.03750	1.04000	1.04250	1.05000	1.06000
2	0.52261	0.52830	0.53020	0.53210	0.53780	0.54544
3	0.35353	0.35864	0.36035	0.36206	0.36721	0.37411
4	0.26903	0.27387	0.27549	0.27712	0.28201	0.28859
5	0.21835	0.22305	0.22463	0.22621	0.23097	0.23740
6	0.18460	0.18921	0.19076	0.19232	0.19702	0.20336
7	0.16051	0.16507	0.16661	0.16815	0.17282	0.17914
8	0.14246	0.14700	0.14853	0.15006	0.15472	0.16104
9	0.12843	0.13297	0.13449	0.13603	0.14069	0.14702
10	0.11723	0.12176	0.12329	0.12483	0.12950	0.13587
11	0.10808	0.11262	0.11415	0.11569	0.12039	0.12679
12	0.10046	0.10501	0.10655	0.10810	0.11283	0.11928
13	0.09403	0.09860	0.10014	0.10170	0.10646	0.11296
14	0.08853	0.09311	0.09467	0.09624	0.10102	0.10758
15	0.08377	0.08838	0.08994	0.09152	0.09634	0.10296
16	0.07961	0.08424	0.08582	0.08741	0.09227	0.09895
17	0.07595	0.08061	0.08220	0.08380	0.08870	0.09544
18	0.07271	0.07740	0.07899	0.08061	0.08555	0.09236
19	0.06981	0.07453	0.07614	0.07776	0.08275	0.08962
20	0.06722	0.07196	0.07358	0.07522	0.08024	0.08718
21	0.06487	0.06965	0.07128	0.07293	0.07800	0.08500
22	0.06275	0.06756	0.06920	0.07086	0.07597	0.08305
23	0.06081	0.06565	0.06731	0.06899	0.07414	0.08128
24	0.05905	0.06392	0.06559	0.06728	0.07247	0.07968
25	0.05743	0.06233	0.06401	0.06571	0.07095	0.07823
26	0.05594	0.06087	0.06257	0.06428	0.06956	0.07690
27	0.05456	0.05953	0.06124	0.06297	0.06829	0.07570
28	0.05329	0.05830	0.06001	0.06175	0.06712	0.07459
29	0.05211	0.05715	0.05888	0.06063	0.06605	0.07358
30	0.05102	0.05609	0.05783	0.05960	0.06505	0.07265
31	0.05000	0.05510	0.05686	0.05864	0.06413	0.07179
32	0.04905	0.05418	0.05595	0.05774	0.06328	0.07100
33	0.04816	0.05332	0.05510	0.05691	0.06249	0.07027
34	0.04732	0.05252	0.05431	0.05613	0.06176	0.06960
35	0.04654	0.05177	0.05358	0.05541	0.06107	0.06897
36	0.04580	0.05107	0.05289	0.05473	0.06043	0.06839
37	0.04511	0.05041	0.05224	0.05410	0.05984	0.06786
38	0.04446	0.04979	0.05163	0.05350	0.05928	0.06736
39	0.04384	0.04921	0.05106	0.05294	0.05876	0.06689
40	0.04326	0.04866	0.05052	0.05242	0.05828	0.06646

Annuity Factors (in arrears) $\frac{q^n(q-1)}{q^n-1} = \frac{i\cdot(1+i)^n}{(1+i)^n-1}$

n	**i**					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
1	1.07000	1.08000	1.09000	1.10000	1.12000	1.12500
2	0.55309	0.56077	0.56847	0.57619	0.59170	0.59559
3	0.38105	0.38803	0.39505	0.40211	0.41635	0.41993
4	0.29523	0.30192	0.30867	0.31547	0.32923	0.33271
5	0.24389	0.25046	0.25709	0.26380	0.27741	0.28085
6	0.20980	0.21632	0.22292	0.22961	0.24323	0.24668
7	0.18555	0.19207	0.19869	0.20541	0.21912	0.22260
8	0.16747	0.17401	0.18067	0.18744	0.20130	0.20483
9	0.15349	0.16008	0.16680	0.17364	0.18768	0.19126
10	0.14238	0.14903	0.15582	0.16275	0.17698	0.18062
11	0.13336	0.14008	0.14695	0.15396	0.16842	0.17211
12	0.12590	0.13270	0.13965	0.14676	0.16144	0.16519
13	0.11965	0.12652	0.13357	0.14078	0.15568	0.15950
14	0.11434	0.12130	0.12843	0.13575	0.15087	0.15475
15	0.10979	0.11683	0.12406	0.13147	0.14682	0.15076
16	0.10586	0.11298	0.12030	0.12782	0.14339	0.14739
17	0.10243	0.10963	0.11705	0.12466	0.14046	0.14451
18	0.09941	0.10670	0.11421	0.12193	0.13794	0.14205
19	0.09675	0.10413	0.11173	0.11955	0.13576	0.13993
20	0.09439	0.10185	0.10955	0.11746	0.13388	0.13810
21	0.09229	0.09983	0.10762	0.11562	0.13224	0.13651
22	0.09041	0.09803	0.10590	0.11401	0.13081	0.13512
23	0.08871	0.09642	0.10438	0.11257	0.12956	0.13392
24	0.08719	0.09498	0.10302	0.11130	0.12846	0.13287
25	0.08581	0.09368	0.10181	0.11017	0.12750	0.13194
26	0.08456	0.09251	0.10072	0.10916	0.12665	0.13113
27	0.08343	0.09145	0.09973	0.10826	0.12590	0.13042
28	0.08239	0.09049	0.09885	0.10745	0.12524	0.12980
29	0.08145	0.08962	0.09806	0.10673	0.12466	0.12925
30	0.08059	0.08883	0.09734	0.10608	0.12414	0.12876
31	0.07980	0.08811	0.09669	0.10550	0.12369	0.12833
32	0.07907	0.08745	0.09610	0.10497	0.12328	0.12795
33	0.07841	0.08685	0.09556	0.10450	0.12292	0.12762
34	0.07780	0.08630	0.09508	0.10407	0.12260	0.12732
35	0.07723	0.08580	0.09464	0.10369	0.12232	0.12706
36	0.07672	0.08534	0.09424	0.10334	0.12206	0.12683
37	0.07624	0.08492	0.09387	0.10303	0.12184	0.12662
38	0.07580	0.08454	0.09354	0.10275	0.12164	0.12644
39	0.07539	0.08419	0.09324	0.10249	0.12146	0.12628
40	0.07501	0.08386	0.09296	0.10226	0.12130	0.12613

Annuity Factors (in arrears) $\frac{q^n(q-1)}{q^n-1} = \frac{i\cdot(1+i)^n}{(1+i)^n-1}$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
41	0.04271	0.04814	0.05002	0.05192	0.05782	0.06606
42	0.04219	0.04765	0.04954	0.05146	0.05739	0.06568
43	0.04170	0.04719	0.04909	0.05102	0.05699	0.06533
44	0.04123	0.04675	0.04866	0.05061	0.05662	0.06501
45	0.04079	0.04634	0.04826	0.05022	0.05626	0.06470
46	0.04036	0.04595	0.04788	0.04985	0.05593	0.06441
47	0.03996	0.04558	0.04752	0.04950	0.05561	0.06415
48	0.03958	0.04523	0.04718	0.04917	0.05532	0.06390
49	0.03921	0.04489	0.04686	0.04886	0.05504	0.06366
50	0.03887	0.04457	0.04655	0.04856	0.05478	0.06344
51	0.03853	0.04427	0.04626	0.04828	0.05453	0.06324
52	0.03822	0.04399	0.04598	0.04801	0.05429	0.06305
53	0.03791	0.04371	0.04572	0.04776	0.05407	0.06287
54	0.03763	0.04345	0.04547	0.04752	0.05386	0.06270
55	0.03735	0.04320	0.04523	0.04729	0.05367	0.06254
56	0.03708	0.04297	0.04500	0.04708	0.05348	0.06239
57	0.03683	0.04274	0.04479	0.04687	0.05330	0.06225
58	0.03659	0.04253	0.04458	0.04668	0.05314	0.06212
59	0.03636	0.04232	0.04439	0.04649	0.05298	0.06199
60	0.03613	0.04213	0.04420	0.04631	0.05283	0.06188
61	0.03592	0.04194	0.04402	0.04614	0.05269	0.06177
62	0.03571	0.04176	0.04385	0.04598	0.05255	0.06166
63	0.03552	0.04159	0.04369	0.04583	0.05242	0.06157
64	0.03533	0.04143	0.04354	0.04568	0.05230	0.06148
65	0.03515	0.04127	0.04339	0.04554	0.05219	0.06139
66	0.03497	0.04112	0.04325	0.04541	0.05208	0.06131
67	0.03480	0.04098	0.04311	0.04529	0.05198	0.06123
68	0.03464	0.04084	0.04299	0.04516	0.05188	0.06116
69	0.03449	0.04071	0.04286	0.04505	0.05179	0.06110
70	0.03434	0.04058	0.04275	0.04494	0.05170	0.06103
71	0.03419	0.04046	0.04263	0.04483	0.05162	0.06097
72	0.03405	0.04035	0.04252	0.04473	0.05154	0.06092
73	0.03392	0.04024	0.04242	0.04464	0.05146	0.06087
74	0.03379	0.04013	0.04232	0.04455	0.05139	0.06082
75	0.03367	0.04003	0.04223	0.04446	0.05132	0.06077
76	0.03355	0.03993	0.04214	0.04438	0.05126	0.06072
77	0.03343	0.03984	0.04205	0.04430	0.05120	0.06068
78	0.03332	0.03975	0.04197	0.04422	0.05114	0.06064
79	0.03322	0.03966	0.04189	0.04415	0.05108	0.06061
80	0.03311	0.03958	0.04181	0.04408	0.05103	0.06057

Annuity Factors (in arrears) $\frac{q^n(q-1)}{q^n-1} = \frac{i\cdot(1+i)^n}{(1+i)^n-1}$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
41	0.07466	0.08356	0.09271	0.10205	0.12116	0.12601
42	0.07434	0.08329	0.09248	0.10186	0.12104	0.12589
43	0.07404	0.08303	0.09227	0.10169	0.12092	0.12579
44	0.07376	0.08280	0.09208	0.10153	0.12083	0.12571
45	0.07350	0.08259	0.09190	0.10139	0.12074	0.12563
46	0.07326	0.08239	0.09174	0.10126	0.12066	0.12556
47	0.07304	0.08221	0.09160	0.10115	0.12059	0.12549
48	0.07283	0.08204	0.09146	0.10104	0.12052	0.12544
49	0.07264	0.08189	0.09134	0.10095	0.12047	0.12539
50	0.07246	0.08174	0.09123	0.10086	0.12042	0.12535
51	0.07229	0.08161	0.09112	0.10078	0.12037	0.12531
52	0.07214	0.08149	0.09103	0.10071	0.12033	0.12527
53	0.07200	0.08138	0.09094	0.10064	0.12030	0.12524
54	0.07186	0.08127	0.09087	0.10059	0.12026	0.12522
55	0.07174	0.08118	0.09079	0.10053	0.12024	0.12519
56	0.07162	0.08109	0.09073	0.10048	0.12021	0.12517
57	0.07151	0.08101	0.09067	0.10044	0.12019	0.12515
58	0.07141	0.08093	0.09061	0.10040	0.12017	0.12514
59	0.07132	0.08086	0.09056	0.10036	0.12015	0.12512
60	0.07123	0.08080	0.09051	0.10033	0.12013	0.12511
61	0.07115	0.08074	0.09047	0.10030	0.12012	0.12509
62	0.07107	0.08068	0.09043	0.10027	0.12011	0.12508
63	0.07100	0.08063	0.09040	0.10025	0.12010	0.12507
64	0.07093	0.08058	0.09036	0.10022	0.12009	0.12507
65	0.07087	0.08054	0.09033	0.10020	0.12008	0.12506
66	0.07081	0.08050	0.09031	0.10019	0.12007	0.12505
67	0.07076	0.08046	0.09028	0.10017	0.12006	0.12505
68	0.07071	0.08043	0.09026	0.10015	0.12005	0.12504
69	0.07066	0.08040	0.09024	0.10014	0.12005	0.12504
70	0.07062	0.08037	0.09022	0.10013	0.12004	0.12503
71	0.07058	0.08034	0.09020	0.10012	0.12004	0.12503
72	0.07054	0.08031	0.09018	0.10010	0.12003	0.12503
73	0.07050	0.08029	0.09017	0.10010	0.12003	0.12502
74	0.07047	0.08027	0.09015	0.10009	0.12003	0.12502
75	0.07044	0.08025	0.09014	0.10008	0.12002	0.12502
76	0.07041	0.08023	0.09013	0.10007	0.12002	0.12502
77	0.07038	0.08021	0.09012	0.10007	0.12002	0.12501
78	0.07036	0.08020	0.09011	0.10006	0.12002	0.12501
79	0.07034	0.08018	0.09010	0.10005	0.12002	0.12501
80	0.07031	0.08017	0.09009	0.10005	0.12001	0.12501

Annuity Factors (in arrears) $\frac{q^n(q-1)}{q^n-1} = \frac{i\cdot(1+i)^n}{(1+i)^n-1}$

n	i					
	0.03	**0.0375**	**0.04**	**0.0425**	**0.05**	**0.06**
85	0.03265	0.03922	0.04148	0.04377	0.05080	0.06043
90	0.03226	0.03892	0.04121	0.04353	0.05063	0.06032
95	0.03193	0.03867	0.04099	0.04333	0.05049	0.06024
100	0.03165	0.03847	0.04081	0.04317	0.05038	0.06018
105	0.03141	0.03830	0.04066	0.04304	0.05030	0.06013

Annuity Factors (in arrears) $\frac{q^n(q-1)}{q^n-1} = \frac{i\cdot(1+i)^n}{(1+i)^n-1}$

n	i					
	0.07	**0.08**	**0.09**	**0.10**	**0.12**	**0.125**
85	0.07022	0.08012	0.09006	0.10003	0.12001	0.12501
90	0.07016	0.08008	0.09004	0.10002	0.12000	0.12500
95	0.07011	0.08005	0.09003	0.10001	0.12000	0.12500
100	0.07008	0.08004	0.09002	0.10001	0.12000	0.12500
105	0.07006	0.08002	0.09001	0.10000	0.12000	0.12500

Appendix B

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